Journal of Mathematical Economics 47 (2011) 417–424

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Multilateral bargaining and Walrasian equilibrium☆

Antonio Penta

Department of Economics, University of Wisconsin - Madison, William H. Sewell Social Science... 1180 Observatory Drive, 53706 Madison, WI, United States

ARTICLE INFO

Article history:
Received 21 January 2011
Accepted 28 March 2011
Available online 30 May 2011

Keywords:
Bargaining games
Multilateral bargaining
Walrasian equilibrium

ABSTRACT

A class of bargaining games in which agents bargain over prices and maximum trading constraints is considered: It is proved that all the Stationary Subgame Perfect Equilibria of these games implement Walrasian allocations as the bargaining frictions vanish. The result holds for any number of agents and is robust to different specifications of the bargaining process.

1. Introduction

Since the early works on marketgames by Shubik (1973), Shapley and Shubik (1977) and Postlewaite and Schmeidler (1978), a large literature has studied the non-cooperative foundations of Walrasian equilibrium. More recently, the development of the theory of strategic bargaining, pioneered by Stahl (1972) and Rubinstein (1982), has motivated the investigation of the foundations of the competitive equilibrium within the context of *strategic bargaining games*[1]: Rather than positing abstract price mechanisms or fictitious auctioneers that deliver the market equilibrium, strategic bargaining games provide a more natural representation of the agents' interaction. More importantly, bargaining games are also more suitable to model economies in which trade is decentralized.[2] The central idea explored by the existing literature is that the competitive outcome should emerge in economies with a large number of agents: For this reason, most works studied economies with an infinite number of agents.[3] The important task of extending the argument to finite economies, possibly letting the number of agents grow large, has proved difficult: Only recently did Gale and Sabourian (2005) provide strategic bargaining foundations to the competitive hypothesis in the context of a single-good (or Marshallian) economy.[4]

To the best of my knowledge, only two contributions have studied strategic bargaining foundations for *finite* Walrasian economies: Yildiz (2005) and Dàvila and Eeckhout (2008) consider pure exchange economies with two agents and an arbitrary number of goods. Analyzing different bargaining procedures, they both provide a particularly striking result: The equilibria of their games yield Walrasian outcomes as the two players become infinitely patient. In Yildiz (2005) it is shown that a bargaining procedure *à la Rubinstein*, in which agents make alternating proposals of *allocations*, yields non-Walrasian outcomes. In contrast, a bargaining procedure in which proposals consist of *price vectors* can implement Walrasian outcomes. Dàvila and Eeckhout (2008) proved Yildiz's conditions to be generically violated in the space of economies; they recovered the competitive result adopting a different bargaining procedure: The two players make alternating offers of *prices* and a *maximum trading constraint*; if the responder agrees, he can demand any trade consistent with the constraints he has agreed upon.[5]

In the economies considered by Yildiz (2005) and Dàvila and Eeckhout (2008), the only possible pairwise meeting also coincides with the grand-coalition of the economy itself. Clearly, this is a special feature of the two-agent case, and it is not clear whether and how their results extend to economies with an arbitrary number of agents. The combined results of Yildiz (2005) and Dàvila and Eeckhout (2008) also point out the sensitivity of

☆ I am indebted to, Jan Eeckhout, George Mailath and Alvaro Sandroni. I also thank two anonymous referees, participants to the 4th Annual *CARESS-Cowles Conference*, Penn EconTheory-Lunch and the VIII SAET Conference in Kos. Special thanks go to Dave "Il Capo" Cass, a mentor and a friend.

E-mail address: apenta@ssc.wisc.edu.

1 See Osborne and Rubinstein (1990) for a survey of this literature.

2 See e.g. Rubinstein and Wolinsky (1985, 1990) and Gale (1986a,b, 1987).

3 The works by Gale (1986a,b) are the classic reference. See also McLennan and Sonnenschein (1991) and more recently Dagan et al. (2000) and Kunimoto and Serrano (2004).

4 For a thorough account of the literature, and a discussion of the issues raised by the finite number of agents, see Gale (2000).

5 The importance of maximum trading constraints for the case of axiomatic bargaining was analyzed by Binmore (1987) first.

0304-4068/$ – see front matter © 2011 Published by Elsevier B.V.
doi:10.1016/j.jmateco.2011.03.002

the competitive outcome to the specification of the bargaining process. This observation motivates further questions concerning the robustness of the results to different specifications of the bargaining protocol.

This paper generalizes Dàvila and Eeckhout's (2008) results to economies with an arbitrary number of agents and to different bargaining processes: It is proved that, as the bargaining frictions vanish, the "stationary subgame perfect equilibria" of a class of bargaining games implement Walrasian allocations in economies with an arbitrary number of agents and commodities. The class of games under consideration encompasses all the bargaining procedures of alternating offers in which the proposer announces prices and maximum trading constraints, in which responses are sequential, trade occurs upon unanimous acceptance, and the continuation game in case of rejection does not depend on the actions previously taken by the players. If an agreement is reached, the proposer acts as the residual claimant of a centralized market: Responders simultaneously choose their demands, subject to the budget and maximum trading constraints, and the market is cleared by the proposer at the announced prices.[6]

Thus, provided that offers are made of prices and maximum trading constraints, the competitive result is robust to details of the bargaining process such as differences in players' discount factors and the process according to which the proposer is selected. Also, the argument does not require a large economy, or an approximation of it such as a *replica* economy: The competitive result holds for any number of agents. This suggests that the details of the bargaining process may play a crucial role in determining the competitive outcome, independent of the role played by the number of agents, which has been the main focus of the literature on non-cooperative bargaining foundations of Walrasian equilibrium.

The rest of the paper is organized as follows: Section 2 introduces the economy, and the basic notation; Section 3 contains the description of the class of bargaining games and discusses the solution concept. Section 4 contains the analysis and main results of the paper. Section 5 concludes.

2. The economy

A pure exchange economy is defined as a tuple $\mathcal{E} = \langle I, r, (X_i, e_i, u_i)_{i\in I}\rangle$: $I = \{1, \ldots, n\}$ is the set of agents, indexed by $i \in I$; $r \in \mathbb{R}^C_{++}$ denotes the total endowments of the C commodities in the economy. For each agent i, preferences are described by a utility function $u_i : \mathbb{R}^C \to \mathbb{R}$. Agent i's consumption possibility set is $X_i \subseteq \mathbb{R}^C_+$. Each agent is endowed with a bundle of goods $e_i \in X_i$ such that $\forall i, 0 \ll e_i \ll r$ and $\sum_{i\in I} e_i = r$.[7] We assume, without loss of generality, that $u_i(e_i) = 0$ for all i.[8] Allocations are denoted by $x = (x_i)_{i\in I} \in \mathbb{R}^{nC}_+$, where for each i, $x_i = \left(x_i^1, \ldots, x_i^C\right) \in \mathbb{R}^C_+$ is the consumption bundle of agent i. An allocation $(x_i)_{i\in I}$ is *feasible* if $\sum_{i\in I} x_i = r$ and $x_i \in X_i$ for each i. X denotes the set of feasible allocations:

$$X = \left\{x \in \mathbb{R}^{nC} : x_i \in X_i \text{ for all } i, \text{ and } \sum_{i\in I} x_i = r\right\}.$$

Prices are denoted by $p \in \mathbb{R}^C_{++}$. The set of Pareto Efficient Allocations is denoted by X^{PE}:

$$X^{\mathrm{PE}} := \left\{x \in X : \nexists x' \in X \text{ such that } \left(u_i\left(x_i'\right)\right)_{i\in I} > \left(u_i\left(x_i\right)\right)_{i\in I}\right\}.$$

Definition 1. The set of *Walrasian Allocations* of an economy $\mathcal{E}$, X^*, is a subset of X satisfying: $\forall x \in X^*$, $\exists p \in \mathbb{R}^C_{++}$ s.t. for each $i \in I$,

$$x_i \in \arg\max_{y_i\in X_i} u_i(y_i) \quad \text{s.t. } p(y_i - e_i) \le 0. \tag{P.1}$$

3. The bargaining game

In this section the bargaining procedure in Dàvila and Eeckhout (2008) is adapted to the case of an economy with an arbitrary number of agents, and is generalized to a wide class of bargaining processes.

Bargaining process. A bargaining process for economy $\mathcal{E}$ is defined by a tuple $\mathcal{S} = (S, \sigma, \pi)$ such that: S is a finite state space; σ is a homogeneous Markov chain taking values in S, with particular realizations $(\sigma_0, \sigma_1, \ldots)$; $\pi : S \to \mathcal{P}(I)$ is a function mapping from the set of states, S, to the set of permutations on I, $\mathcal{P}(I)$. For each $s \in S, \pi(s) = (\pi_1(s), \ldots, \pi_n(s))$ identifies the order in which agents move in state s. We refer to the agent $\pi_1(s) \equiv a(s)$ as the *auctioneer* in state s; the other agents are the *traders*. The selected *auctioneer* $a(s) \in I$ announces a price vector p, and a vector $q = \left(q_j\right)_{j\neq a(s)} \in \mathbb{R}^{C(n-1)}$, where q_j represents player j's maximum excess demand (hereafter, we will refer to q_j as j's *maximum trading constraint, MTC*). The remaining agents $j \in I \setminus \{a(s)\}$ play sequentially, $\pi_2(s)$ moving first, and so on, until $\pi_n(s)$: They may either accept (action "Y") or reject (action "N"). If everybody accepts, trade can take place in the centralized market at the price p announced by the *auctioneer*, subject to the traders' MTCs $\left(q_j\right)_{j\neq a(s)}$: Traders simultaneously choose excess demands $\left(z_j\right)_{j\neq a(t)}$ s.t. $z_j \in B_j\left(p, q_j\right)$, where

$$B_j\left(p, q_j\right) := \left\{z \in \mathbb{R}^C : z \le q_j \text{ and } pz \le 0\right\}. \tag{1}$$

The aggregate excess demand $\sum_{j\neq a(s)} z_j$ is cleared by the auctioneer, acting as the residual claimant of the market. For the game to be well-defined, the auctioneer must be capable of clearing all the individual demands consistent with the individual budget constraints. To this purpose, a restriction on the auctioneers' announcements is imposed: $\forall s, \sum_{j\neq a(s)} q_j \le e_{a(s)}$. That is, the auctioneer must be able to clear the maximum number of quantities that are allowed to be traded. After trade has taken place, agents leave the market and consume the bundle of goods they own.

If any player rejects, no trade occurs and the system moves to the next period according to the process σ.

The definition of the set of histories and of players' strategies is straightforward but notationally cumbersome, and so is omitted. Strategy profiles are denoted by $f = (f_1, \ldots, f_n)$, f_i being i's strategy and f_{-i} his opponents'.

Payoffs. Agents discount time: for each $i \in I$, let $\delta_i \in (0, 1)$ denote agent i's discount factor, and $\delta = (\delta_1, \ldots, \delta_n)$ denote the profile of discount factors. The payoff in case of perpetual disagreement is assumed to be zero. If agreement occurs at period t and agent i holds bundle x_i after trade, he consumes it and derives a utility of $u_i(x_i)$. Player i's payoff for this outcome is therefore $\delta_i^t u_i(x_i)$.

Bargaining game. For given economy $\mathcal{E}$ and bargaining process $\mathcal{S}$, let $\Gamma(\delta)$ denote the game described above with players' discount factors $\delta \in (0, 1)^n$. Similarly, $\Gamma(\mathbf{1})$ denotes the game induced by $\mathcal{E}$ and $\mathcal{S}$ when players are infinitely patient.

The setup above leaves a lot of freedom to the specification of S and σ, and therefore a large class of bargaining processes are consistent with the present framework. The maintained assumptions used for the main result will be discussed in Section 4. For now, notice that the framework introduced thus

[6] Some alternative specifications are considered in Section 5.

[7] The notation adopted for vector inequalities is as follows: "$\ll$" represents a strict inequality for all components; "$<$" allows equality for some component, but not all; "$\le$" means "$<$" or "$=$".

[8] This normalization is done so that the disagreement outcome, set to zero in Section 3, is payoff equivalent to the autarchy consumption.

far encompasses all the bargaining procedures that use price-posting and maximum trading constraints, in which trade occurs upon unanimous acceptance, responses are sequential, and the continuation game in case of rejection does not depend on the actions previously taken by the players. It includes, for example, deterministic processes of alternating offers, with each agent making the offer an arbitrary number of consecutive periods; or a game in which at every period, each player is equally likely to occupy any positions in the order of moves, and so on. The only important restriction is that the transition probabilities only depend on the current state, not on the previous history or on the players' actions (e.g. the identity of who has rejected).

3.1. Solution concept

The solution concept adopted here, *Stationary Subgame Perfect Equilibrium* (SSP), is a selection from the Subgame Perfect Equilibria which imposes the additional restriction that equilibrium strategies must be *stationary*, i.e. the continuation after any history of length t is completely determined by the realized state σ_t.[9]

Definition 2. A strategy profile f is a *Stationary Subgame Perfect Equilibrium (SSP)* if it is a Subgame Perfect Equilibrium and if f_i is stationary for each i.

In a SSP, in each period, players' behavior only conditions on the current state and the moves previously made in that period. Note in particular that stationarity does not require a player as auctioneer to always make the same offer. That is, player i always makes the same offer when selected as the auctioneer in state s, but not necessarily the same offer at two states s, s' such that $a(s) = a(s') = i$. Similarly, for each s, $\pi_k(s)$'s response after a particular sequence of responses of agents $\pi_l(s)$, $l = 2, \ldots, k-1$ is always the same; but this does not mean that agents always respond in the same way to agent i's proposal: For instance, suppose that there exist two (distinct) states s and s' at which i is the auctioneer and j the first respondent (i.e. $a(s) = a(s') = i$ and $\pi_2(s) = \pi_2(s') = j$), and suppose that i's strategy is such that i makes the same offer (p, q) at s and s'. The stationarity restriction does not rule out the possibility that j accepts (p, q) at s but not at s'. Similar freedom is left by the stationarity assumption to the behavior of later respondents.

4. Maintained assumptions and results

The main result is based on two sets of assumptions, respectively on the economy $\mathcal{E}$ and the bargaining process $\mathcal{S}$.

Maintained assumptions on $\mathcal{E}$: For each i, $X_i \subseteq \mathbb{R}^C_+$ is convex and u_i satisfies the following:

- (E.1): u_i is differentiably strictly increasing.
- (E.2): u_i is differentiably strictly quasi-concave on an open set $G_i \supseteq X_i$.
- (E.3): $\{x_i \in G_i : u_i(x_i) \geq u_i(e_i)\} \cap bd(X_i) = \emptyset$.
- (E.4): u_i is *strongly concave*, in the sense that

$$\det\left\{2D^2 u_i(x) + \left[\sum_{k=1}^{n} D_{ikj} u_i(x_i)\left[x_k - e_{i,k}\right]\right]_{ij}\right\}$$

does not change sign.

Assumptions (E.1-3) are standard in general equilibrium theory, and are imposed to guarantee a well-behaved pure-exchange economy: (E.1-2) are the standard assumptions for the differentiable versions of the welfare theorems[10]; (E.3) requires that the upper contour sets of the initial endowments do not intersect the boundaries of the consumption sets. For $p \in \mathbb{R}^C_{++}$, assumptions (E.3) allows to dispense with the compactness of X_i (or with the assumption $X_i = \mathbb{R}^C_+$): Under (E.3), for $p \in \mathbb{R}^C_{++}$, the optimization problem (P.1) has a solution. (E.3) also rules out corner solutions and guarantees that equilibrium allocations do not lie on the boundary of the set of feasible allocations (i.e. $X^* \subseteq \operatorname{int} X$). Assumptions (E.1-3) together also imply that the set of Pareto efficient allocations coincides with the set of "pairwise efficient" allocations, that is the set of allocations for which no pair of agents can induce a Pareto improvement through bilateral exchange (Proposition 1, p. 47 Gale, 2000).

Assumption (E.4) was introduced by Dàvila and Eeckhout (2008), and is needed for the analysis of the bargaining game. It guarantees that the offer curves have no inflexion points, and it is satisfied whenever the substitution effect dominates the income effect. As in Dàvila and Eeckhout (2008), assumption (E.4) will be used to prove that agreement in the bargaining game occurs with no delay (Proposition 1).

Maintained assumptions on $\mathcal{S}$:

- (S.1): S is a finite set of states.
- (S.2): For each $i \in I$, $\exists s \in S : a(s) = i$;
- (S.3): every state $s \in S$ communicates with every other state $s' \in S$.

Assumption (S.1) will be used to prove existence of SSP in the game with discounting (Proposition 1). Assumptions (S.2-3) together imply that, from any initial condition, each agent is selected as the auctioneer in finite time with probability one. This imposes a lower bound on the bargaining power of each agent. No restrictions are imposed on the relative frequency with which agents are selected as proposers.

4.1. Main result

In this section it will be proved that, under the maintained assumptions (E.1-4) and (S.1-3), the *SSP outcomes* of $\Gamma(\delta)$ converge to *Walrasian allocations* as $\delta \to \mathbf{1}$.

The basic strategy of the proof parallels its counterpart in Dàvila and Eeckhout (2008), adapting it to the general class of bargaining protocols considered here. The argument of the proof proceeds as follows: first, it is shown that if all players are impatient ($\delta \ll \mathbf{1}$), an SSP exists and all SSP are *with immediate acceptance*, i.e. such that agreement occurs at all states (Proposition 1); second, it is proved that the correspondence of the SSP of the game is upper hemicontinuous in δ on $(0, 1]^n$ (Proposition 2): hence, if SSP are *with immediate acceptance* for $\delta \ll \mathbf{1}$, also their limits are *SSP with immediate acceptance* at $\delta = \mathbf{1}$; Finally, it is shown that at $\delta = \mathbf{1}$, all the *SSP with immediate acceptance* induce Walrasian allocations (Proposition 3), thus completing the proof of the main result (Proposition 4).

4.1.1. Proof of the main result

Consider the utility possibility set of the economy, defined as

$$\mathcal{U} := \left\{v \in \mathbb{R}^n : \exists x \in X \text{ s.t. } u(x) = v\right\}.$$

Each strategy profile f induces an *outcome* of the bargaining game, defined by a pair $\left(\tau^f, \eta^f\right)$, where τ^f is a *stopping time* and η^f is a random variable taking values in $\mathcal{U}$. The stopping time τ^f denotes the time at which agreement occurs, while η^f denotes the utilities

[9] Hence, for f to be an equilibrium, it must be immune to all possible unilateral deviations, stationary and non-stationary.

[10] (E.2) is often strengthened to requiring differentiable strict quasi-concavity on all $\mathbb{R}^C$ (see for instance Mas-Colell et al. (1995), Section 16.F).

agents derive from the consumption bundles they own at that period: Notice that because of the underlying stochastic process σ, $\left(\tau^f, \eta^f\right)$ are in general non-degenerate random variables, even if f is a profile of pure strategies. The dependence of the outcome on the strategy profile will be suppressed, and the outcome simply denoted by (τ, η), when its meaning is clear from the context.

For any pair (τ, η), and for each state s,

$$\mathbb{E}\left[\delta^\tau \eta | \sigma_0 = s\right] = \left(\mathbb{E}\left[\delta_i^\tau \eta_i | \sigma_0 = s\right]\right)_{i \in I}$$

is the profile of expected utilities of the outcome (τ, η) when the initial state is s.

An outcome (τ, η) is *stationary* if there exists a measurable subset $S^* \subseteq S$ and a measurable function $\xi : S^* \to X$ such that: (i) $\sigma_t \notin S^*$ for all $t = 0, 1, \ldots, \tau - 1$; (ii) $\sigma_\tau \in S^*$; (iii) $\eta = u\left(\xi\left(\sigma_\tau\right)\right)$. In words, a *stationary outcome* can be characterized by a pair (ξ, S^*) such that $S^* \subseteq S$ is the set of states in which agreement occurs, and the function ξ delivers the resulting allocation at each state. Using the latter condition, for any strategy profile f that induces a stationary outcome, we may define the value function of f at state s, $v^f(s) = \mathbb{E}\left[\delta^\tau u\left(\xi\left(\sigma_\tau\right)\right) | \sigma_0 = s\right]$. Clearly, an SSP must induce a *stationary outcome*. Hence, the subsequent analysis will focus on *stationary outcomes* only.

The next definition introduces *SSPs with no delay*:

Definition 3. An *SSP with immediate acceptance* (or *with no delay*) is an SSP in which agreement occurs in all states. Formally: f is an SSP with immediate acceptance if it induces a stationary outcome (ξ, S^*) s.t. $S^* = S$.

An *SSP with immediate acceptance* can be characterized by a tuple $(p^s, q^s)_{s \in S}$, where for each s, (p^s, q^s) is the offer made by $a(s)$ in state s. Traders $j \neq a(s)$ accept and choose consumption bundles $x_j(p^s, q^s)$ such that:

$$x_j\left(p^s, q^s\right) \in \arg\max_{x_j \in X_j} u_j\left(x_j\right) \tag{2}$$
$$\text{s.t.} \quad \begin{aligned} & p^s\left[x_j - e_j\right] \leq 0 \\ & \left[x_j - e_j\right] \leq q_j^s. \end{aligned}$$

This condition derives from the definition of subgame perfection: Once an agreement is reached, subgame perfection requires that each responder solves the optimization problem defined in (2). By stationarity of strategies, $a(s)$ always offers the same (p^s, q^s) in state s in a SSP. Furthermore, under the maintained assumptions (E.1-2), $x_j(p^s, q^s)$ is uniquely determined for each j and s. Thus, an *SSP with immediate acceptance* can be completely characterized by a tuple $(p^s, q^s)_{s \in S}$, which in turn determines a tuple $(x^s)_{s \in S}$ of corresponding allocations.

Given this observation, in an *SSP with immediate acceptance*, at each state s the proposer $a(s)$ optimizes under the constraint that none of the traders $j \neq a(s)$ has an incentive to deviate, that is $\forall s \in S$:

$$\left(p^s, q^s\right) \in \arg\max_{(p,q)} u_{a(s)}\left(r - \sum_{j \neq a(s)} x_j\left(p^s, q^s\right)\right) \tag{3}$$
$$\text{s.t.:} \begin{Bmatrix} u_j\left(x_j\right) \geq \delta_j \mathbb{E}\left[u_j\left(x_j^{\sigma_1}\right) | \sigma_0 = s\right] \\ \text{for } x_j\left(p^s, q^s\right) \text{ defined as in (2)} \end{Bmatrix}_{j \neq a(s)}.$$

The first constraint in (3) is a "no rejection" condition, necessary for responders to accept the offer rather than delaying the agreement and moving to the next period in state σ_1. The second constraint is simply the subgame perfection condition discussed above.

It is worth to point out that once an agreement is reached, players do not face a strategic situation anymore: They are simply left with the solution of the optimization problem in (2), and they behave as price takers. The agents' strategic behavior is confined to the responses and the offers, i.e. the bargaining process. Once the latter is over, agents do not behave strategically.[11]

Since, upon agreement, responders are free to choose any consumption bundle consistent with the constraints in (2), in any *SSP with immediate acceptance* $(p^s, q^s)_{s \in S}$ the induced allocations $(x^s)_{s \in S}$ must be such that, for each s and $j \neq a(s)$,

$$Du_j\left(x_j^s\right)\left[x_j^s - e_j\right] \geq 0.$$

The inequality is strict if the maximum trading constraint q_j^s is binding in (2), i.e. if the bundle x_j demanded at p^s relaxing the MTC is such that $\left(x_j - e_j\right) > q_j^s$. Furthermore, since the proposer at s chooses the tuple $\left(q_j^s\right)_{j \neq a(s)}$, conditional on the responders accepting the offer, $a(s)$ can induce *any* allocation that satisfies $Du_{-a(s)}\left(x_{-a(s)}^s\right)\left[x_{-a(s)}^s - e_{-a(s)}\right] \geq 0$ simply making the MTCs tighter. Hence, an *SSP with immediate acceptance* can be characterized by allocation offers $\left(\left(x_j^s\right)_{j \in I}\right)_{s \in S}$, where $\left(x_j^s\right)_{j \in I}$ is the allocation offered by $a(s)$ at s, such that[12]:

$$\forall s \in S:$$
$$\left(x_l^s\right)_{l \in I} \in \arg\max_{(x_l)_{l \in I} \in X} u_{a(s)}\left(x_{a(s)}\right) \tag{4}$$
$$\text{s.t.:} \begin{Bmatrix} u_j\left(x_j\right) \geq \delta_j \mathbb{E}\left[u_j\left(x_j^{\sigma_1}\right) | \sigma_0 = s\right] \\ Du_j\left(x_j\right)\left[x_j - e_j\right] \geq 0 \end{Bmatrix}_{j \neq a(s)}.$$

Impatient players. In this section it is proved that in all the SSP of the game with impatient players, agreement occurs with no delay, and that an SSP exists. The analysis is conducted in the space of utilities, exploiting techniques similar to Merlo and Wilson's (1995): *SSP payoffs* are characterized as the fixed points of a map from a space of measurable functions to itself. Such measurable functions represent the utility profiles induced by profiles of stationary strategies.

Proposition 1. *If $\delta \ll \mathbf{1}$, in any SSP of the game, agreement occurs with no delay. Furthermore, an SSP exists.*

Proof. Consider the utility possibility set of the economy

$$\mathcal{U} := \left\{v \in \mathbb{R}^n : \exists x \in X \text{ s.t. } u(x) = v\right\}.$$

An outcome (τ, η) is *stationary* if there exists a measurable subset $S^* \subseteq S$ and a measurable function $\mu : S^\mu \to \mathcal{U}$ such that: (i) $\sigma_t \notin S^\mu$ for all $t = 0, 1, \ldots, \tau - 1$; (ii) $\sigma_\tau \in S^\mu$; (iii) $\eta = \mu(\sigma_\tau)$. Using the latter condition, for any stationary outcome, we may define, for all s, $v^\mu(s) = \mathbb{E}\left[\delta^\tau \mu\left(\sigma_\tau\right) | \sigma_0 = s\right]$. Let $\mathcal{V}^n$ denote the set of bounded and measurable functions $v : S \to \mathbb{R}^n$.

Lemma 1. *If (μ, S^μ) is a stationary outcome, then v^μ is the unique function in $\mathcal{V}^n$ such that:*

$$v^\mu(s) = \mu(s) \quad \text{for all } s \in S^\mu, \text{ and}$$
$$v^\mu(s) = \mathbb{E}\left[\delta v\left(\sigma_1\right) | \sigma_0 = s\right] \quad \text{for all } s \in S \setminus S^\mu.$$

Proof of Lemma 1. Given (μ, S^μ), define $V : \mathcal{V}^n \to \mathcal{V}^n$ s.t. $\forall v \in \mathcal{V}^n$,

$$V(v)(s) = \begin{cases} \mu(s) & \text{if } s \in S^\mu \\ \mathbb{E}\left[\delta v\left(\sigma_1\right) | \sigma_0 = s\right] & \text{otherwise.} \end{cases}$$

The lemma is established if v^μ is the unique solution in $\mathcal{V}^n$ to the fixed-point problem $V(v) = v$.

11 This is a consequence of the structure of the game, not an assumption on agents' behavior.

12 Cf. Lemma A1 in Dàvila and Eeckhout (2008).

Step 1: $V(\cdot)$ is a contraction. Let $\|\cdot\|$ denote the supnorm on $\mathbb{R}^n$, and $\|\cdot\|_\infty$ the supnorm on $\mathcal{V}^n$. Let $v, v' \in \mathcal{V}^n$. Then, if $s \in S^\mu$, $\|V(v)(s) - V(v')(s)\| = 0$; if $s \in S \setminus S^\mu$,

$$\begin{aligned}\|V(v)(s) - V(v')(s)\| &= \|\mathbb{E}[\delta[v(\sigma_1) - v'(\sigma_1)]|\sigma_0 = s]\| \\ &\leq \beta\|\mathbb{E}[v(\sigma_1) - v'(\sigma_1)]|\sigma_0 = s\| \\ &\leq \beta\|v - v'\|_\infty\end{aligned}$$

where $\beta := \max\{\delta_i : i \in I\}$. Hence, $\exists\beta \in (0,1) : \|V(v) - V(v')\|_\infty \leq \beta\|v - v'\|_\infty$. Since $\mathcal{V}^n$ is a complete metric space, Banach's theorem implies that $V(\cdot)$ has a unique fixed point.

Step 2: $V(v^\mu) = v^\mu$. Define the stopping time for agreement starting at period $t = 1$ as τ_1, such that: $\sigma_{\tau_1} \in S^\mu$ and $\sigma_t \notin S^\mu$ for $t = 1, \ldots, \tau_1 - 1$. Then, for any $s \in S$,

$$\begin{aligned}v^\mu(s) &= \mathbb{E}[\delta^\tau v(\sigma_\tau)|\sigma_0 = s] \\ &= \mathbb{E}[\delta^{\tau_1 - 1} v(\sigma_{\tau_1})|\sigma_1 = s].\end{aligned}$$

If $s \in S^\mu$, $V(v^\mu)(s) = v^\mu(s)$ simply by definition. If $\sigma_0 = s \in S \setminus S^\mu$, then $\tau = \tau_1$, so:

$$\begin{aligned}V(v^\mu)(s) &= \mathbb{E}[\delta v(\sigma_1)|\sigma_0 = s] \\ &= \mathbb{E}[\delta\mathbb{E}[\delta^{\tau_1 - 1} v(\sigma_{\tau_1})|\sigma_1]|\sigma_0 = s] \\ &= \mathbb{E}[\delta\mathbb{E}[\delta^{\tau - 1} v(\sigma_\tau)|\sigma_1]|\sigma_0 = s] \\ &= \mathbb{E}[\delta^\tau v(\sigma_\tau)|\sigma_0 = s] \\ &= v^\mu(s).\end{aligned}$$

This completes the proof of Lemma 1. □

As discussed above, a necessary condition for allocation x to be an SSP outcome is that $Du_j(x_j)[x_j - e_j] \geq 0$ for each j. Hence, for the analysis of SSP payoffs we can restrict attention to the utility space

$$\mathcal{U}^* := \{v \in \mathbb{R}^n : \exists x \in X \text{ s.t. } u(x) = v \text{ and } Du(x)[x - e] \geq 0\}.$$

Any SSP determines a *stationary outcome* (τ, η) with η taking values in $\mathcal{U}^*$. Under assumptions (E.1-4) the set $\mathcal{U}^*$ is compact and strictly convex, and $0 \in \mathcal{U}^*$. Hence, for any (τ, η) such that the image of η lies in $\mathcal{U}^*$, it must be $\mathbb{E}[\delta^\tau\eta|\sigma_0 = s] \in \mathcal{U}^*$ for all s.

Let $\mathcal{W}$ be the set of measurable functions $w : S \to \mathcal{U}^*$. For each agent i, define the function $\varphi_i^* : \mathbb{R}^n \to \mathbb{R}$ such that, for all $d \in \mathbb{R}^n$,

$$\varphi_i^*(d) := \begin{cases} 0 & \text{if } \nexists v \in \mathcal{U}^* : v_{-i} \geq d_{-i} \\ \max\{v_i : v \in \mathcal{U}^*, \text{ and } v_{-i} \geq d_{-i}\} & \text{otherwise.}\end{cases}$$

Under the maintained assumptions (E.1-3), function φ_i^* is well defined and continuous for each i.

Define the operator $E : \mathcal{W} \to \mathcal{W}$ such that for $w \in \mathcal{W}$,

$$E_i(w)(s) = \begin{cases} \max\{\varphi_i^*(\mathbb{E}[\delta w(\sigma_1)|\sigma_0 = s]); & \\ \quad \mathbb{E}[\delta_i w_i(\sigma_1)|\sigma_0 = s]\} & \text{if } i = a(s) \\ \mathbb{E}[\delta_i w_i(\sigma_1)|\sigma_0 = s] & \text{otherwise.}\end{cases}$$

Clearly, E is a continuous map.

Lemma 2. *v^* is a SSP payoff if and only if $E(v^*) = v^*$.*

Proof of Lemma 2. ($\Rightarrow$) Let v^* be a SSP payoff. Fix $s \in S$ and let $i = a(s)$. If agreement does not occur at s, it must be $v^*(s) = E[\delta v^*(\sigma_1)|\sigma_0 = s]$. Now, consider an alternative proposal $v \in U^*$ at some s. If $v_j < E[\delta v^*(\sigma_1)|\sigma_0 = s]$ for some j, the proposal is rejected; if $v_j \geq E[\delta_j v_j^*(\sigma_1)|\sigma_0 = s]$ for all j, proposal v is accepted in a SSP. Hence, a payoff maximizing proposer would obtain $\varphi_i^*(\mathbb{E}[\delta v^*(\sigma_1)|\sigma_0 = s])$ from any proposal that is accepted. Since i can always induce a rejection, for agreement to occur it must be $\varphi_i^*(\mathbb{E}[\delta v^*(\sigma_1)|\sigma_0 = s]) \geq E[\delta_i v_i(\sigma_1)|\sigma_0 = s]$. In other words, if v^* is a SSP payoff, it satisfies

$$v^*(s) = \begin{cases} \mathbb{E}[\delta v^*(\sigma_1)|\sigma_0 = s] & \text{if } s \in S \setminus S^\mu \\ (\max\{\varphi_{a(s)}^*(\mathbb{E}[\delta v^*(\sigma_1)|\sigma_0 = s]); & \\ \quad \mathbb{E}[\delta_{a(s)} v_{a(s)}^*(\sigma_1)|\sigma_0 = s]\}, & \\ \quad \mathbb{E}[\delta_{-a(s)} v_{-a(s)}^*(\sigma_1)|\sigma_0 = s]) & \text{if } s \in S^\mu \end{cases}$$

which clearly satisfies $E(v^*) = v^*$.

($\Leftarrow$) From the one-shot deviation principle and the stationarity of equilibrium strategies, any deviation that induces a rejection when an acceptance is due would yield the continuation payoff. From the definition of E, in a fixed point the continuation is never strictly greater than the value at any given s. Hence, a fixed point of E can be sustained by a SSP of the game. This proves the lemma. □

Remark 1. By construction, if v^* is a fixed point of E, then $v^* \in bd(\mathcal{U}^*)$.

We can now prove the first part of Proposition 1: From the strict convexity of $\mathcal{U}^*$, if $\delta \ll \mathbf{1}$, for any outcome (τ, η), $E[\delta^\tau\eta|\sigma_0 = s] \in \mathcal{U}^*$ for all s. Let v^* be a SSP payoff, and suppose that there exists a state s in which agreement is not reached. Then, $v^*(s) = E[\delta^\tau\eta|\sigma_0 = s] \in \text{int}(\mathcal{U}^*)$. But this is inconsistent with v^* being a fixed point of E (Remark 1).

To prove existence, it is sufficient to prove the following:

Lemma 3. *$\langle\mathcal{W}, \|\cdot\|_\infty\rangle$ is a compact, convex, complete metric space.*

Proof of Lemma 3. *Convexity*: let $w, w' \in \mathcal{W}$, $w \neq w'$. For $\alpha \in (0,1)$, let $w^\alpha(s) = \alpha w(s) + (1 - \alpha)w'(s)$ for all s. Since U^* is convex, clearly $w^\alpha : S \to \mathcal{U}^*$. It is clearly measurable, hence $w^\alpha \in \mathcal{W}$.

Compactness: for any sequence $\{w^\nu\}_{\nu\in\mathbb{N}} \subseteq \mathcal{W}$, for each $s \in S$, $\{w^\nu(s)\}_{\nu\in\mathbb{N}}$ is a sequence in $\mathcal{U}^*$, hence with a limit $\bar{w}(s) \in \mathcal{U}^*$. Hence, $\{w^\nu\}_{\nu\in\mathbb{N}} \to \bar{w}$ pointwise. Hence in the supnorm.

Completeness: Since it is a compact subset of a complete metric space, it is also complete. □

Since $E : \mathcal{W} \to \mathcal{W}$ is continuous and $\langle\mathcal{W}, \|\cdot\|_\infty\rangle$ is a non-empty, compact, convex, subset of a linear metric space, the existence of a fixed point follows from Schauder's Theorem. This completes the proof of Proposition 1. □

The convergence result. From Proposition 1, if players discount, in any SSP an agreement is reached at any state with no delay. Therefore, the SSP outcomes can be represented by measurable functions $y : S \to X$, assigning a feasible allocation to each state. Let Y be the set of such measurable functions. For any initial state s, $y(s)$ is the allocation induced by the acceptance of $a(s)$'s offer. Similar to the above, SSP allocations are fixed points of the operator $\rho : Y \to Y$, defined as:

$$\rho(y;\delta)(s) = \arg\max_{x\in X} u_{a(s)}(x_{a(s)})$$

$$\text{s.t. for all } j \neq a(s), \quad \begin{array}{l} u_j(x_j) \geq \mathbb{E}[\delta_j u_j(y_j(\sigma_1))|\sigma_0 = s] \\ Du_j(x_j)[x_j - e_j] \geq 0.\end{array}$$

With a slight abuse of notation, let us consider the operator ρ as a function of δ and define the correspondence $\Lambda : [0,1]^n \rightrightarrows Y$ such that

$$\Lambda(\delta) = \{y \in Y : y \in \rho(y;\delta)\}.$$

$\Lambda(\delta)$ is the set of fixed points of ρ, as a function of δ.

Proposition 2. *$\Lambda(\delta)$ is an upper hemicontinuous correspondence.*

Proof. Since (from Berge's Maximum Theorem) ρ is u.h.c. in δ, also $\Lambda(\delta)$ is u.h.c. (this follows from Lemma A3 in Dàvila and Eeckhout, 2008). □

Infinitely patient players. From Propositions 1 and 2 we know that the limit of the SSP, as $\delta \to \mathbf{1}$, are with immediate acceptance. The last step of the analysis is then to prove that all such equilibria implement Walrasian allocations. To this end, notice that if $\delta = \mathbf{1}$, under assumptions (S.1-3) the proposer's problem in (4) can be rewritten as:

$$\forall s \in S:$$
$$\left(x_l^s\right)_{l\in I} \in \arg\max_{(x_l)_{l\in I}\in X} u_{a(s)}\left(x_{a(s)}\right) \tag{5}$$
$$\text{s.t.:} \left\{\begin{array}{c} Du_{ij}\left(x_j\right)\left[x_j - e_j\right] \geq 0 \\ u_j\left(x_j\right) \geq \max\left\{u_j\left(x_j^{s'}\right) : s' \in S\right\}\end{array}\right\}_{j\neq a(s)}$$

The reason is that under (S.1-3), from any state s, any state s' is reached in finite time with probability one. With infinitely patient players then the incentive compatibility constraint is that above because a player would reject as long as the utility he obtains in state s is lower than what he would obtain in any other state.

Proposition 3. *If $\delta_i = 1$ for every $i \in I$, the outcome $\left(x_l^*\right)_{l\in I}$ of an SSP with immediate acceptance is a Walrasian allocation.*

Proof. The proof is completed by the following lemmata.

Lemma 4. *If $\delta = \mathbf{1}$, in a SSP with immediate acceptance $\left(x_l^s\right)_{l\in I} = \left(x_l^*\right)_{l\in I}$ for all $s \in S$.*

Proof of Lemma 4. Suppose, by means of contradiction, that there exist $s \neq s'$ such that $\left(x_l^s\right)_{l\in I} \neq \left(x_l^{s'}\right)_{l\in I}$. Notice that for each s, constraints $u_j\left(x_j\right) \geq \max\left\{u_j\left(x_j^{s'}\right) : s' \in S\right\}$ must be binding for every j in equilibrium, which implies that in equilibrium, $u_j\left(x_j^s\right) = u_j\left(x_j^{s'}\right)$ for each $j \in I$ and all $s, s' \in S$. From strict convexity of preferences, if $u_j\left(x_j^s\right) = u_j\left(x_j^{s'}\right)$ and $x_j^s \neq x_j^{s'}$, $\left(x_l^s\right)_{l\in I}$ is inefficient, hence (by assumptions (E.1-3)) pairwise inefficient. Since $\left(x_l^s\right)_{l\in I}$ is an SSP-outcome, it satisfies $Du_l\left(x_l^s\right)\left[x_l^s - e_l\right] \geq 0$ for all l. Inefficiency then implies that $Du_j\left(x_j^s\right)\left[x_j^s - e_j\right] > 0$. Continuity of $Du_j(\cdot)$ (assumption (E.2)) and the pairwise inefficiency of $\left(x_l^s\right)_{l\in I}$ then imply that for some $k \neq j$, there exists a transfer z from j to k such that $Du_j\left(x_j^s - z\right)\left[x_j^s - z - e_j\right] \geq 0$ and

$$u_k\left(x_k^s + z\right) > u_k\left(x_k^s\right)$$
$$u_j\left(x_j^s - z\right) \geq u_j\left(x_j^s\right).$$

Then, if $k = a(s)$, (i.e. k is making the offer at s), inducing the responders to accept $\left(\left(x_i^s\right)_{i\neq k,j}, \left(x_j^s + z\right)\right)$, and consuming $\left(x_k^s + z\right)$ wold be a profitable deviation, because $\left(x_k^s + z\right)$ is strictly preferred to x_k^s. If $k \neq a(s)$ (i.e. k is one of the responders), by assumptions (S.1-3) a state s' in which $a\left(s'\right) = k$ would be reached in finite time with probability one. Since there is no discounting, rejecting until s' is reached and then make the offer above would be a profitable deviation for k. Thus, $\left(x_l^s\right)_{l\in I} = \left(x_l^*\right)_{l\in I}$ for all $s \in S$. This completes the proof of Lemma 4. □

Lemma 5. *At $\left(x_l^*\right)_{l\in I}$, $Du_i\left(x_i^*\right)\left[x_i^* - e_i\right] = 0$ for each i.*

Proof of Lemma 5. Given Lemma 4, in a SSP with immediate acceptance it must be the case that, for each i, and $j \neq k \neq i \neq j$, $\left(x_k^*\right)_{k\neq i}$ solves:

$$\left(x_k^*\right)_{k\neq i} \in \arg\max_{(x_k)_{k\neq i}} u_i\left(r - \sum_{k\neq i} x_k\right)$$
$$\text{s.t.:} \left\{\begin{array}{c} Du_k\left(x_k\right)\left[x_k - e_k\right] \geq 0 \\ u_k\left(x_k\right) \geq u_k\left(x_i^*\right)\end{array}\right\}_{k\neq i}.$$

Now, suppose that $J = \left\{j \in I : Du_j\left(x_j^*\right)\left[x_j^* - e_j\right] > 0\right\} \neq \emptyset$, then $\left(x_k^*\right)_{k\neq i}$ also solves

$$\left(x_k^*\right)_{k\neq i} \in \arg\max_{(x_k)_{k\neq i}} u_i\left(r - \sum_{k\neq i} x_k\right)$$
$$\text{s.t. } u_j\left(x_j\right) \geq u_j\left(x_j^*\right) \quad \text{for all } j \in J$$
$$\left\{\begin{array}{c} Du_k\left(x_k\right)\left[x_k - e_k\right] \geq 0 \\ u_k\left(x_k\right) \geq u_k\left(x_i^*\right)\end{array}\right\}_{k\notin J\cup\{i\}}.$$

Notice that $\left[x_i^* - e_i\right] = \left[\sum_{k\neq i} e_k - \sum_{k\neq i} x_k^*\right]$ and $\sum_{j\in J}[x_j^* - e_j] = \left[\sum_{k\notin J} e_k - x_k^*\right]$. Fix $j \in J$. From efficiency, under (E.1-3), $\forall l \neq j, \exists \gamma_{jl} > 0 : Du_j\left(x_j^*\right) = \gamma_{jl} Du_l\left(x_l^*\right)$. Let $\gamma_{ji} = \min\left\{\gamma_{jl} : l \neq j\right\}$. Adding up the constraints for j's optimization problem

$$\begin{aligned}
0 &\leq \sum_{k\neq j} Du_k\left(x_k^*\right)\left[x_k^* - e_k\right] \\
&= Du_j\left(x_j^*\right)\left[\sum_{k\neq j}\frac{1}{\gamma_{jk}}\left[x_k^* - e_k\right]\right] \\
&= Du_j\left(x_j^*\right)\left[\frac{1}{\gamma_{ji}}\left[x_i^* - e_i\right] + \sum_{k\neq i,j}\frac{1}{\gamma_{jk}}\left[x_k^* - e_k\right]\right] \\
&= Du_j\left(x_j^*\right)\left[\frac{1}{\gamma_{ji}}\left[\sum_{k\neq i} e_k - \sum_{k\neq i} x_k^*\right] + \sum_{k\neq i,j}\frac{1}{\gamma_{jk}}\left[x_k^* - e_k\right]\right] \\
&= Du_j\left(x_j^*\right)\left[\frac{1}{\gamma_{ji}}\left[e_j - x_j^*\right] + \sum_{k\neq i,j}\left(\frac{1}{\gamma_{jk}} - \frac{1}{\gamma_{ji}}\right)\left[x_k^* - e_k\right]\right] \\
&= Du_j\left(x_j^*\right)\left[\frac{1}{\gamma_{ji}}\left[e_j - x_j^*\right] + \sum_{k\in J\setminus\{j,i\}}\left(\frac{1}{\gamma_{ji}} - \frac{1}{\gamma_{jk}}\right)\left[e_k - x_k^*\right]\right. \\
&\qquad \left. + \sum_{k\notin J\cup\{i\}}\left(\frac{1}{\gamma_{jk}} - \frac{1}{\gamma_{ji}}\right)\left[x_k^* - e_k\right]\right] \\
&= \frac{1}{\gamma_{ji}} Du_j\left(x_j^*\right)\left[e_j - x_j^*\right] + \sum_{k\in J\setminus\{j,i\}}\left(\frac{\gamma_{jk}}{\gamma_{ji}} - 1\right) Du_k\left(x_k^*\right)\left[e_k - x_k^*\right] \\
&\qquad + \sum_{k\notin J\cup\{i\}}\left(1 - \frac{\gamma_{jk}}{\gamma_{ji}}\right) Du_k\left(x_k^*\right)\left[x_k^* - e_k\right]
\end{aligned}$$

which yields the desired contradiction: The absurd hypothesis implies that the first term and the first summation are negative, while the terms in the last summation are all negative by construction, for $\frac{\gamma_{jk}}{\gamma_{ji}} > 1$ and $Du_k\left(x_k^*\right)\left[x_k^* - e_k\right] \geq 0$ for $k \notin J$. This completes the proof of Lemma 5. □

From Lemma 5, it suffices to set $p^* = Du_1\left(x_1^*\right)$, to obtain, for each i:

$$x_i^* \in \arg\max_{x_i} u_i\left(x_i\right)$$
$$\text{s.t.: } p^* x_i = p^* e_i.$$

Hence, $\left(x_l^*\right)_{l\in I}$ is a Walrasian allocation, which completes the proof of Proposition 3. □

Main result. We can now conclude proving the main proposition:

Proposition 4. *If the maintained assumptions* (E.1-4) *and* (S.1-3) *are satisfied, the SSP outcomes of $\Gamma(\delta)$ converge to Walrasian allocations as $\delta \to \mathbf{1}$.*

Proof. From Proposition 1 we know that if $\delta \ll \mathbf{1}$, an SSP exists and all SSP are *with immediate acceptance*; Proposition 2 implies that the limits of the SSP as $\delta \to \mathbf{1}$ are *SSP with immediate acceptance* of Γ (**1**); Proposition 3 shows that SSP with immediate acceptance of Γ (**1**) induce Walrasian allocations. Hence, combining these results, the SSP of Γ (δ) converge to Walrasian allocations as $\delta \to \mathbf{1}$. □

4.2. A converse result

In this section, a converse to Proposition 3 is provided: If agents are infinitely patient ($\delta = \mathbf{1}$), every Walrasian allocation can be sustained as the outcome of a *SSP with immediate acceptance*.

Proposition 5. *Let* $\left(p^*, \left(x_l^*\right)_{l\in I}\right)$ *be a WE of the economy* $\mathcal{E}$. *Then, under the maintained assumptions* (E.1-3) *and* (S.1-3)*, there is an SSP with immediate acceptance of* Γ (**1**) *with outcome* $\left(x_l^*\right)_{l\in I}$.

Proof. Given the WE $\left(p^*, \left(x_l^*\right)_{l\in I}\right)$, consider the following stationary strategy profile: $\forall i \in I$, whenever i makes a proposal (i.e. for all $s \in a^{-1}(i)$) he offers $\left(p^i, q^i\right)$ such that:

- $p^i = p^*$; $\left(q_j^i\right)_{j\neq i}$ are slack, i.e. $q_j^i \geq z_j\left(p^*\right)$, where

  $$z_j\left(p^*\right) = \arg\max_{z_j\in\mathbb{R}^C} u_j\left(e_j + z_j\right)$$

  $$\text{s.t.:}\quad p^* z_j \leq 0.$$

- Whenever i is responding, he accepts any offer $\left(p', q'\right)$ such that $u_i\left(x_i\left(p', q'\right)\right) \geq u_i\left(x_i^*\right)$.

Clearly, the outcome of this strategy profile, starting from any subgame, is $\left(x_l^*\right)_{l\in I}$. To check that this strategy is indeed an SSP, we need to consider all possible unilateral deviations, stationary and non-stationary.[13] Notice that if i deviates at any point of the game, he may only induce one of the following types of outcomes.

1. Agreement is never reached.
2. At a later stage in the game, i offers $\left(p', q'\right)$ such that $\forall j \neq i, u_j\left(x_j\left(p', q'\right)\right) \geq u_j\left(x_j^*\right)$ and it is accepted. (That $u_j\left(x_j\left(p', q'\right)\right) \geq u_j\left(x_j^*\right)$ for each $j \neq i$ is a necessary condition for acceptance, given the opponents' equilibrium strategy profile.)
3. At a later stage in the game, i accepts the offer $\left(p^*, q^j\right)$ made by $j \neq i$, which yields the same outcome $\left(x_l^*\right)_{l\in I}$.

None of these cases corresponds to a profitable deviation for player i: Outcome (1) yields a payoff of $0 = u_i\left(e_i\right) \leq u_i\left(x_i^*\right)$. Hence, inducing perpetual disagreement cannot be a profitable deviation. Consider outcome (2) now: Since $\left(x_l^*\right)_{l\in I}$ is efficient, and $u_j\left(x_j\left(p', q'\right)\right) \geq u_j\left(x_j^*\right)$ for all $j \neq i$, it cannot be that $u_i\left(r - \sum_{j\neq i} x_j\left(p', q'\right)\right) > u_i\left(x_i^*\right)$. Hence this outcome cannot be preferred to $\left(x_l^*\right)_{l\in I}$ by i. Case (3) induces the same outcome as the candidate equilibrium, therefore it cannot be a profitable deviation either. □

Since assumptions (E.1-3) guarantee the existence of a Walrasian equilibrium (WE), the result indirectly proves existence of SSP for the case $\delta = \mathbf{1}$. (Notice that (E.4) is not needed for Proposition 5.)

Corollary 1. *Under the maintained assumptions* (E.1-3) *and* (S.1-3)*, there exists an SSP of* Γ (**1**).

5. Concluding remarks

In this model agents bargain over prices and maximum trading constraints. If an agreement is reached, trade occurs in a *centralized* way. The results of Dàvila and Eeckhout (2008) for two-agent economies are generalized to economies with an arbitrary (finite) number of agents, and to different bargaining procedures. It is proved that, as the bargaining frictions vanish, the stationary subgame perfect equilibria implement Walrasian allocations in economies with an arbitrary number of agents and commodities. To the best of my knowledge, this is the first work that provides strategic bargaining foundations in such general environments.

A remarkable aspect of the result is that it does not require a large economy, or an approximation of that such as a *replica* economy: The result holds for any finite number of agents.

On the role of the bargaining procedure. The role that different bargaining procedures may play in providing strategic bargaining foundations of Walrasian equilibrium is a thoroughly unexplored question: The existing literature in this research agenda has considered almost exclusively a specific bargaining procedure (namely, a *take-it or leave-it* exchange proposal *à la Rubinstein*), and has focused mainly on the role played by the number of agents in the economy.[14] The findings of this paper, Yildiz (2005) and Dàvila and Eeckhout (2008) combined, suggest that a careful analysis of alternative bargaining protocols may prove fruitful for this research agenda, and a promising direction for future research.

Centralized vs. decentralized trade. In economies with more than two agents, trade can be centralized or decentralized. In this paper, trade is centralized, and it occurs in a one-shot exchange. This is an obvious limitation of the model, but it allows to focus on the properties of the bargaining procedure, abstracting from the complications specific to trade being decentralized.

Gale and Sabourian (2005) obtained the competitive result for *decentralized Marshallian markets* (i.e. single-good economies) with a finite number of agents. In their environment, agents' gains from trade can be exhausted in a single pairwise exchange.[15] Thus, from the point of view of each agent, there is only one relevant exchange. On the contrary, considering *Walrasian markets* (i.e. with an arbitrary number of goods), the agents' gains from trade cannot in general be exhausted in any given pairwise meeting: Each agent has to go through a sequence of bilateral exchanges before the gains from trade are exhausted, and therefore he must also keep track of the other agents' sequences of exchanges. For this reason, *decentralized Walrasian markets* are significantly more complex than *decentralized Marshallian markets* (e.g. Gale and Sabourian, 2005) and than *centralized Walrasian markets* (as in this paper): In the latter two, each agent's trade occurs in a one-shot exchange.

In a companion paper (Penta, 2007) an alternative specification of the model is explored, in which agents are randomly matched in pairs, and trade occurs through a sequence of bilateral exchanges. Within each pair, agents adopt a bargaining procedure similar to that analyzed above. A partial competitive result has been obtained thus far: If an economy has a sufficiently large number of agents and the initial allocation is sufficiently close to the Pareto set, then Walrasian allocations can be achieved in a decentralized way, through a sequence of pairwise matchings in which agents bargain and trade.[16]

On the robustness result. In this paper the robustness of the results obtained from the bargaining procedure is also addressed: The results hold for a class of games that encompasses all the bargaining procedures of alternating offers in which the proposer

[13] Notice also that with no discounting the *one-shot deviation principle* does not apply. Hence all possible deviations must be considered, one-shot or not.

[14] See for instance Gale (2000) and references therein. Also, Gale and Sabourian (2005).

[15] This property also holds in the related papers of Rubinstein and Wolinsky (1985, 1990), Gale (1987) and Sabourian (2004).

[16] How close the endowments need to be to the Pareto set depends on the degree of substitutability of goods: with more substitutability the competitive outcome may be obtained (as $\delta \to 1$) for a larger set of initial conditions. The trade-off is related to the possibility of strategically manipulating the terms of trade when either big trades are involved, or the marginal rates of substitution change a lot. Making trades smaller, or reducing the effect of trades on the marginal rates of substitution, reduces the extent to which the terms of trade can be manipulated.

announces prices and maximum trading constraints, in which trade occurs upon unanimous acceptance, the continuation game in case of rejection does not depend on the actions previously taken by the players, and responses are sequential. The latter qualification is not essential: If traders are allowed to respond simultaneously, the competitive result can be obtained applying to a refinement of SSP that ensures agents do not play weakly dominated strategies. Such refinement involves trembles in the players' responses: If in a SSP trader k is rejecting an offer at some history, all traders $j \neq k$ are indifferent between rejecting and accepting that offer, because k's rejection makes j's actions at that history all outcome-equivalent. For this reason, if players respond simultaneously, we may have for instance an SSP in which everybody rejects every offer. The consideration of trembles in the players' responses rules out this sort of equilibria based on players' coordinations on a rejection. No trembles in agents' offers or demands are necessary for the result.[17]

On the Stationarity restriction. It is important to emphasize that the restriction to stationary equilibria is a strong one. Other than the simplicity of the analysis, the general argument in favor of stationary strategies is that they entail relatively simpler behavior, and would therefore be chosen by somewhat boundedly rational agents. But this argument does not seem convincing in general games.[18] Recently, Sabourian (2004) and Gale and Sabourian (2005) have made precise the sense in which boundedly rational agents would play stationary strategies in the equilibria of their models. This allows them to overcome the difficulties arisen in Rubinstein and Wolinsky (1990) without assuming away the use of non-stationary strategies. The present paper did not focus on these issues of complexity, and the stationarity of equilibrium strategies is simply assumed. Chatterjee and Sabourian (2000) obtain stationarity of equilibrium strategies in multi-person bargaining games through the introduction of complexity costs. Whether the adoption of SSP in the present setting may be supported by a similar notion of bounded rationality is an interesting question left to future research.

Implementation of Walrasian allocations. The results of this paper may be interpreted as one of limit implementation of the Walrasian equilibrium correspondence (the limit concerning the agents' discount factors). From this point of view, the present work is also related to Hurwicz (1979), who studied the problem of implementing Walrasian allocations from a mechanism design perspective: In the mechanism constructed by Hurwicz (1979), players simultaneously announce pairs of vectors $(p_i, y_i), i = 1, \ldots, n$, where y_i can be interpreted as the proposed trade in non-numeraire goods for trader i, and p_i can be interpreted as the price vector for non-numeraire goods proposed by i for use in the other traders' budget constraints. Interestingly, the message spaces in Hurwicz (1979) are thus similar to the proposals available to the auctioneer in the bargaining game above. The price vector p^* at which agents trade results from a function of the announced $(p_1, \ldots, p_n)$; such function is specified by the mechanism, and involves a term that penalizes players that unilaterally deviate from profiles of uniform announcements. This way, everyone announces the same p^* in equilibrium. Furthermore, under the properly designed scheme of transfers, such equilibrium p^* will also be a Walrasian price.

One thing is worth pointing out though. The mechanism design approach *à la* Hurwicz, as well as the market-game literature *à la* Shapley and Shubik, are very different in spirit from the literature (more relevant to the present paper) on strategic bargaining foundations. This is because, similar to market games, it is difficult to interpret the function that maps agents' actions to the market price in Hurwicz (1979) as resulting from an underlying bargaining procedure among the agents, which is instead the main emphasis of the literature on strategic bargaining foundations.

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[17] A proof is available from the author.
[18] Cf. Mailath and Samuelson (2006, ch. 5).