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## HIGHER ORDER UNCERTAINTY AND INFORMATION: STATIC AND DYNAMIC GAMES

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## HIGHER ORDER UNCERTAINTY AND INFORMATION: STATIC AND DYNAMIC GAMES

BY ANTONIO PENTA<sup>1</sup>

Weinstein and Yildiz (2007) have shown that in static games, only very weak predictions are robust to perturbations of higher order beliefs. These predictions are precisely those provided by interim correlated rationalizability (ICR). This negative result is obtained under the assumption that agents have no information on payoffs. This assumption is unnatural in many settings. It is therefore natural to ask whether Weinstein and Yildiz's results remain true under more general information structures. This paper characterizes the "robust predictions" in static and dynamic games, under arbitrary information structures. This characterization is provided by an extensive form solution concept: interim sequential rationalizability (ISR). In static games, ISR coincides with ICR and does not depend on the assumptions on agents' information. Hence the "no information" assumption entails no loss of generality in these settings. This is not the case in dynamic games, where ISR refines ICR and depends on the details of the information structure. In these settings, the robust predictions depend on the assumptions on agents' information. This reveals a hitherto neglected interaction between information and higher order uncertainty, raising novel questions of robustness.

**KEYWORDS:** Dynamic games, hierarchies of beliefs, higher order beliefs, information, interim sequential rationalizability, robustness, uniqueness.

### 1. INTRODUCTION

ECONOMIC MODELLING TYPICALLY INVOLVES making common knowledge assumptions. It is therefore natural to ask which predictions of a model retain their validity when such assumptions are relaxed. Recently, Weinstein and Yildiz (2007) have shown that in static games, such predictions are precisely those provided by *interim correlated rationalizability* (ICR; Dekel, Fudenberg, and Morris (2007)). They showed that any ICR action profile can be made uniquely ICR by perturbing agents' higher order beliefs (their *structure theorem*). This implies that any refinement of ICR is not robust, as it rules out outcomes that would be uniquely selected in some arbitrarily close model. Overall, the result has important negative implications: no equilibrium concept delivers robust predictions once common knowledge assumptions are relaxed.

To prove the structure theorem, Weinstein and Yildiz (2007) assumed a *richness condition* on the underlying space of uncertainty. This condition requires

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that for every action of every player there exists a state under which that action is strictly dominant. This means that essentially *all* common knowledge assumptions are relaxed, as under the richness condition it is *not* common knowledge among the players that any action is *not* dominant. Weinstein and Yildiz also assumed that players have no information about payoffs (their own or their opponents') and that this is common knowledge.

The "no information" assumption is unnatural in many settings. For example, agents competing in an auction may have private information on the value of the object; in bilateral trade environments, it may be natural to assume that traders know their valuation for the good, and sometimes one may wish to maintain that this is common knowledge too (i.e., assume private values). In yet other settings, one may not wish to impose common knowledge that agents have no information, allowing instead agents to be uncertain as to whether their opponents are privately informed or not. For instance, in an auction it may be common knowledge that some agents have information and not others, but the identities of the informed and uninformed may not be common knowledge. It is natural then to ask whether the negative results of Weinstein and Yildiz are specific to the no information assumption, and to investigate what predictions are robust under more general information structures.<sup>2</sup>

In this paper, I characterize the predictions that are robust to the relaxation of common knowledge assumptions, in both static and dynamic games, when higher order beliefs are perturbed in general information structures. To this end, I introduce a new solution concept: interim sequential rationalizability (ISR). ISR is an extensive form solution concept, which coincides with ICR in static games. I show that, irrespective of the information structure, a structure theorem analogous to Weinstein and Yildiz's holds for ISR.

In static games, ISR does not depend on the assumptions on the information structure. Hence, Weinstein and Yildiz's results extend to arbitrary information structures: the no information assumption entails no loss of generality. But this is not true in dynamic games, as ISR depends on the assumptions on agents' information, raising novel questions of robustness.

These results show that, while unimportant in static games, the fine details of informational assumptions may be crucial in dynamic games. This has important implications for applied theory. For instance, Bergemann and Morris (2007) recently argued that in auction problems with interdependent values, ascending auctions can be useful in reducing strategic uncertainty so as to eliminate undesirable outcomes that could arise in their sealed-bid counterparts.

<sup>2</sup>A complementary question would be to consider less demanding robustness tests, in which some common knowledge assumptions are maintained (for instance, that some actions are *not* dominant). In Penta (2011), I studied the structure of ICR on arbitrary spaces of uncertainty, that is, without assuming richness. I showed that the structure theorem remains true under very mild relaxations of the common knowledge assumptions, which reinforces Weinstein and Yildiz's message.

The results of the present paper suggest that the robustness of that insight depends on the details of agents' information about payoffs: for instance, in perturbing agents' higher order beliefs, if we assume that agents have no information about payoffs, all the advantages of the ascending auction disappear. On the contrary, the robustness properties of sealed-bid auctions are not affected by changes in the informational assumptions.

More generally, it can be argued that the no information assumption is too restrictive in dynamic games.<sup>3</sup> The main reason for studying these games is that if agents cannot commit to their strategies, "sequential rationality" considerations impose stronger restrictions than those implied by the normal form approach. But once common knowledge assumptions are relaxed, the very notion of sequential rationality has no bite if agents have no information on payoffs. This is because, with no information, players' beliefs about payoffs after unexpected moves are only restricted by what is commonly known. If all such assumptions are relaxed (e.g., assuming richness), sequential rationality has no bite at zero probability histories, which means that it coincides with "normal form rationality." Notice that this reasoning does not involve higher order beliefs. It follows immediately from the richness and no information assumptions combined.

In seeking to characterize the robust predictions in dynamic games, it is natural to look at extensive form analogues of ICR, that is, dynamic counterparts of the assumptions of rationality and common belief in rationality. The weakest of such counterparts assumes that agents are sequentially rational and that this is common belief *at the beginning of the game* (initial common belief in sequential rationality (ICBSR)). A natural question, therefore, is how to characterize the "strongest robust predictions" that satisfy such a minimal requirement. ISR comprises precisely such a strongest robust solution concept: refinements of ISR (e.g., extensive form rationalizability or perfect Bayesian equilibrium) are not robust. On the other hand, ISR characterizes the behavioral implications of ICBSR. Hence, the strongest robust predictions are also the weakest among those consistent with ICBSR.

Weinstein and Yildiz (2007) also proved that once common knowledge assumptions are relaxed, static games are generically dominance-solvable. This result generalizes an important insight from the literature on (static) global games: the pervasive multiplicity of equilibria that we observe in standard models stems from the high degree of coordination implicit in the common knowledge assumptions. The validity of this insight in dynamic contexts has been questioned by the recent literature on dynamic global games, in which the familiar uniqueness results do not obtain. By proving a generic uniqueness result

<sup>3</sup>Independent work by Chen (2011) extended the structure theorem for ICR to (finite) dynamic games in normal forms, maintaining the no information assumption. The relationship with his work is discussed in Section 4.1.

for ISR, I show in contrast that the same insight extends to dynamic games, irrespective of the information structure.<sup>4</sup>

The rest of the paper is organized as follows. Section 2 introduces the game theoretic framework. Section 3 introduces ISR and some of its properties. Section 4 provides the structure theorem. Section 5 introduces a novel notion of robustness (information invariance) motivated by the role that assumptions on information have in dynamic games. Section 6 concludes.

## 2. GAME THEORETIC FRAMEWORK

The analysis that follows applies to multistage games with observable actions (Fudenberg and Tirole (1991, Sect. 8.2)),<sup>5</sup> which are defined by an *extensive form* that represents agents' possible moves and information about the opponents' moves, a *preference-information structure* (PI structure hereafter) that represents players' information about everyone's payoffs, and a *type space* that represents agents' beliefs.<sup>6</sup> These concepts are formally introduced next.

### 2.1. Extensive Forms

An extensive form is defined by a tuple

$$\Gamma = \langle N, \mathcal{H}, \mathcal{Z}, (A_i)_{i \in N} \rangle,$$

where  $N = \{1, \dots, n\}$  is the set of players and for each player  $i$ ,  $A_i$  is the (finite) set of his possible actions. Histories are finite concatenations of action profiles, and the (finite) set of all possible histories is partitioned into the set of terminal histories  $\mathcal{Z}$  and the set of partial histories  $\mathcal{H}$  (the latter includes the *empty history*  $\phi$ ). As the game unfolds, the partial history  $h$  that has just occurred becomes public information and is perfectly recalled by all players. For each  $h \in \mathcal{H}$  and  $i \in N$ , let  $A_i(h)$  denote the (finite) set of actions available to player  $i$  at history  $h$ , and let  $A(h) = \times_{i \in N} A_i(h)$  and  $A_{-i}(h) = \times_{j \in N \setminus \{i\}} A_j(h)$ .<sup>7</sup> Without loss of generality,  $A_i(h)$  is assumed to be nonempty for each  $h$ : player  $i$  is *inactive* at  $h$  if  $|A_i(h)| = 1$  and he is *active* otherwise. This setup allows finitely

<sup>4</sup>Morris and Shin (2003) surveyed the literature on (static) global games. On dynamic global games, see, for instance, Angeletos, Hellwig, and Pavan (2007), who considered an infinite horizon game, which does not fall within the present setting. Nonetheless, as they discussed (Angeletos, Hellwig, and Pavan (2007, p. 729)), their multiplicity result also holds in the finite horizon version of their model.

<sup>5</sup>The analysis can be easily extended to all finite dynamic games with *perfect recall*, but restricting attention to multistage games with observable actions significantly simplifies the notation.

<sup>6</sup>This terminology is not entirely standard: Bayesian games are usually defined by an extensive form, payoff functions, and a type space. Information and beliefs are often conflated in the type space. The separation proposed here is common in the literature on robust mechanism design (see references in footnote 11).

<sup>7</sup>Formally,  $A_i(h) = \{a_i \in A_i : \exists a_{-i} \in A_{-i} \text{ s.t. } (h, (a_i, a_{-i})) \in \mathcal{H} \cup \mathcal{Z}\}$ .

repeated games as a special case or games with perfect information if  $\mathcal{H}$  is such that only one player is active at each  $h$ . If  $\mathcal{H} = \{\phi\}$ , the game is static.

*Pure strategies* of player  $i$  assign to each partial history  $h \in \mathcal{H}$  an action in  $A_i(h)$ . Let  $S_i$  denote the set of *reduced form strategies* (or *plans of action*) of player  $i$ . Two strategies correspond to the same reduced strategy  $s_i \in S_i$  if and only if they are realization-equivalent to  $s_i$ , that is, they preclude the same collection of histories and for every nonprecluded history  $h$ , they select the same action,  $s_i(h)$ .<sup>8</sup> Each profile  $s$  induces a unique terminal history  $\mathbf{z}(s) \in \mathcal{Z}$ . For each  $h \in \mathcal{H}$ , let  $S_i(h)$  be the set of strategies  $s_i$  that allow  $h$  to be reached (that is, there exists  $s_{-i}$  such that  $h$  is on the path to  $\mathbf{z}(s_i, s_{-i})$ ). Let  $\mathcal{H}(s_i) = \{h \in \mathcal{H} : s_i \in S_i(h)\}$  denote the set of partial histories not precluded by  $s_i$ . Since only reduced strategies are considered here, I omit the term “reduced” in the following discussion.

### 2.2. PI Structures

Players’ payoffs are represented by functions  $u_i : \mathcal{Z} \rightarrow \mathbb{R}$  for each  $i$ . To model situations in which payoffs are not common knowledge, payoff functions are parametrized on a space  $\Theta$ ,  $u_i : \mathcal{Z} \times \Theta \rightarrow \mathbb{R}$ .<sup>9</sup> Elements of  $\Theta$  are referred to as *payoff states*. Players’ information is modelled as an information partition on  $\Theta$ . To avoid unnecessary complications, I focus on partitions with a product structure, so that  $\Theta$  can be written as<sup>10</sup>

$$\Theta = \Theta_0 \times \Theta_1 \times \dots \times \Theta_n.$$

For each  $i = 1, \dots, n$ ,  $\Theta_i$  is the set of player  $i$ ’s *payoff types*.  $\Theta_0$  is referred to as the set of *states of nature*. Sets  $\Theta_k$  ( $k = 0, 1, \dots, n$ ) are assumed to be compact subsets of Euclidean spaces and each  $u_i$  is concave in  $\theta_i$ . Also define  $\Theta_{-i} = \prod_{j \in N \setminus \{i\}} \Theta_j$ , so that  $\Theta = \Theta_0 \times \Theta_i \times \Theta_{-i}$ .

In state  $(\theta_0, \theta_1, \dots, \theta_n)$ , player  $i$ ’s payoff type is  $\theta_i$ , which is observed at the beginning of the game. Hence, payoff types represent agents’ *information* about the payoff state: if  $i$ ’s payoff type is  $\hat{\theta}_i$ ,  $i$  knows that the true state belongs to the set  $\Theta_0 \times \{\hat{\theta}_i\} \times \Theta_{-i}$ . The set  $\Theta_0$  represents the residual uncertainty that is left after pooling everybody’s information.

<sup>8</sup>The standard notion of strategy specifies a player’s behavior also at histories precluded by the strategy itself. In equilibrium, this counterfactual behavior represents the opponents’ beliefs about  $i$  in case he has deviated from the strategy. This paper follows a nonequilibrium approach, and players’ beliefs about the opponents’ behavior are modelled explicitly. We can thus restrict attention to plans of actions.

<sup>9</sup>This representation is without loss of generality. For example, taking  $\Theta \equiv ([0, 1]^n)^\mathcal{Z}$  and letting  $u_i(z, \theta) = \theta(z)$  imposes no restrictions on agents’ payoffs.

<sup>10</sup>The extension to general information partitions is conceptually straightforward, but notationally cumbersome. The same simplifying restriction is common in the literature on mechanism design (e.g., references in footnote 11).

The tuple  $\langle \Theta_0, (\Theta_i, u_i)_{i \in N} \rangle$  thus represents agents' information about payoffs. It is assumed to be common knowledge and is referred to as preference-information structure (PI structure). Here are a few examples of special cases:

(i) If  $\Theta_k$  is a singleton for all  $k = 0, 1, \dots, n$ , then the game has *complete information*.

(ii) If  $\langle \Theta_0, (\Theta_i, u_i)_{i \in N} \rangle$  is such that, for every  $i \in N$ ,  $u_i$  is constant in  $(\theta_0, \theta_{-i})$ , the PI structure is one of *private values*.

(iii) If  $\Theta_i$  is a singleton for every  $i$  (hence  $\Theta \simeq \Theta_0$ ), then agents' have no information about payoffs and this is common knowledge. This is the special case considered by Weinstein and Yildiz (2007) and Chen (2011). (Notice that private values is neither a stronger nor a weaker assumption than no information.)

(iv) If  $\Theta_0$  is a singleton, the PI structure is one of *distributed knowledge*. These structures are common in the literature on robust mechanism design.<sup>11</sup>

(v) The private values and no information assumptions are somewhat extreme and sometimes too restrictive. For instance, it may be desirable to allow agents to have private information without necessarily imposing that this is common knowledge. For example, let  $\Theta_0 = [1, 2]$ ,  $\Theta_i = [0, 3]$ , and payoff functions  $u_i$  be such that for some  $U_i \in [0, 1]^Z$ ,  $u_i(z, \theta) = (\theta_i - \theta_0)U_i(z)$ . Hence, if  $\theta_i > 2$  or  $\theta_i < 1$ , player  $i$  knows his own preferences ( $U_i$  and  $-U_i$ , respectively), but if  $\theta_i \in [1, 2]$ , player  $i$  has essentially no information. In such a PI structure, whether  $i$  is informed is not common knowledge (of course, other common knowledge assumptions are implicit in this example).

### *Information and Beliefs*

Agents also entertain (subjective) beliefs about the components of the state they do not know. For instance, agent  $i$  may assign probability 1 to some strict subset  $E \subset \Theta$ , that is, he is *certain* of  $E$ . Beliefs are introduced in Section 2.3, but it is useful to point out an important difference between information and beliefs (or between knowledge and certainty). First of all, while agents may entertain wrong beliefs (i.e., be certain of false events), information is never false in this model. Player  $i$  knows  $\hat{\theta}_i$  only if the realized state is in  $\Theta_0 \times \{\hat{\theta}_i\} \times \Theta_{-i}$ , that is, if  $\hat{\theta}_i$  is true.<sup>12</sup>

As the game unfolds, agents may change their beliefs about the payoff state in response to observing the opponents' past moves. When both information and beliefs are represented in the same model, it is standard to maintain that

<sup>11</sup>See, for example, Bergemann and Morris (2005, 2009) and Artemov, Kunimoto, and Serrano (2011) for static mechanisms; see Mueller (2010) and Penta (2010a) for dynamic mechanisms.

<sup>12</sup>The difference between knowledge and belief is well understood in epistemological game theory. At a formal level, the difference is the so-called *axiom of truth* (or *axiom of knowledge*), which is satisfied in *knowledge structures*, not in *belief structures*. The axiom of truth represents precisely the idea that " $i$  knows  $E$ " only if " $E$  is true whenever  $i$  knows  $E$ ." (See Osborne and Rubinstein (1994, Chap. 5).)

agents' beliefs never contradict their information.<sup>13</sup> This implies that if  $i$ 's payoff type is  $\theta_i$ , player  $i$ 's beliefs are concentrated on  $\Theta_0 \times \{\theta_i\} \times \Theta_{-i}$  at every history. That is,  $i$  always is certain of  $\theta_i$ . In contrast, initial certainty about elements other than  $\theta_i$  does not imply that such beliefs are maintained later in the game: a player who initially is certain that  $(\theta_0, \theta_{-i}) \in E$  may abandon this belief after observing an unexpected event, such as a zero probability history. Endowing agents with more information therefore restricts the set of feasible beliefs after unexpected events. This is why assumptions on information, which are inconsequential in static games, are crucial in dynamic games.

### 2.3. Exogenous Beliefs

To complete the description of the strategic situation, players' beliefs about what they do not know must be specified. That is, for every  $i$ , his beliefs about  $\Theta_0 \times \Theta_{-i}$  (*first-order beliefs*), his beliefs about  $\Theta_0 \times \Theta_{-i}$  and the opponents' first-order beliefs (*second-order beliefs*), and so on.

Given a PI structure  $\langle \Theta_0, (\Theta_i, u_i)_{i \in N} \rangle$ , players' hierarchies of beliefs are defined as usual (see Mertens and Zamir (1985)): for each  $i \in N$ , let  $Z_i^1 = \Delta(\Theta_0 \times \Theta_{-i})$  denote the set of player  $i$ 's first-order beliefs, and for  $k \geq 1$ , define recursively<sup>14</sup>

$$Z_{-i}^k = \prod_{j \neq i} Z_j^k$$

and

$$Z_i^{k+1} = \left\{ (\pi_i^1, \dots, \pi_i^{k+1}) \in Z_i^k \times \Delta(\Theta_0 \times \Theta_{-i} \times Z_{-i}^k) : \text{marg}_{\Theta_0 \times \Theta_{-i} \times Z_{-i}^k} \pi_i^{k+1} = \pi_i^k \right\}.$$

Agent  $i$ 's first-order beliefs are elements of  $\Delta(\Theta_0 \times \Theta_{-i})$ ; an element of  $\Delta(\Theta_0 \times \Theta_{-i} \times Z_{-i}^{k-1})$  is a  $\Theta$ -based  $k$ -order belief for every  $k > 1$ . The set of (collectively coherent)  $\Theta$ -hierarchies is defined as

$$H_{i,\Theta} = \left\{ (\pi_i^1, \pi_i^2, \dots) \in \left( \prod_{k \geq 1} \Delta(\Theta_0 \times \Theta_{-i} \times Z_{-i}^{k-1}) \right) : (\pi_i^1, \dots, \pi_i^k) \in Z_i^k \ \forall k \geq 1 \right\}.$$

<sup>13</sup>This requirement is both standard and natural. The very notion of information would be problematic if agents' beliefs were not required to be consistent with it. In the epistemic literature, this requirement is one of the axioms that define information in dynamic models of knowledge and belief (see Battigalli and Bonanno (1999) for a survey).

<sup>14</sup>For any set  $X$ ,  $\Delta(X)$  denotes the set of probability distributions over  $X$ , endowed with the topology of weak convergence.



Players’  $\Theta$ -hierarchies are represented by means of type spaces.

DEFINITION 1— $\Theta$ -Based Type Space: A ( $\Theta$ -based) type space is a tuple

$$\mathcal{T} = \langle \Theta_0, (\Theta_i, T_i, \theta_i, \tau_i)_{i \in N} \rangle$$

such that for each  $i \in N$ ,  $T_i$  is a set of types,  $\theta_i: T_i \rightarrow \Theta_i$  is an onto function that assigns to each type a payoff type and  $\tau_i: T_i \rightarrow \Delta(\Theta_0 \times T_{-i})$  assigns to each type a belief about the states of nature and the opponents’ types.<sup>15</sup> We focus here on compact type spaces, in which sets  $T_i$  are compact, and functions  $\tau_i$  and  $\theta_i$  are continuous.

Each type in a type space induces a  $\Theta$ -hierarchy: the first-order beliefs induced by  $t_i \in T_i$  are obtained by the map  $\hat{\pi}_i^1: T_i \rightarrow \Delta(\Theta_0 \times \Theta_{-i})$  defined as follows: for every measurable  $E \subseteq \Theta_0 \times \Theta_{-i}$ ,

$$\hat{\pi}_i^1(t_i)[E] = \tau_i(t_i)[\{(\theta_0, t_{-i}) \in \Theta_0 \times T_{-i} : (\theta_0, \theta_{-i}(t_{-i})) \in E\}].$$

For  $k > 1$ , the induced  $k$ -order beliefs are obtained by mappings  $\hat{\pi}_i^k: T_i \rightarrow \Delta(\Theta_0 \times \Theta_{-i} \times Z_{-i}^{k-1})$ , defined recursively as follows: for every measurable  $E \subseteq \Theta_0 \times \Theta_{-i} \times Z_{-i}^{k-1}$ ,

$$\hat{\pi}_i^k(t_i)[E] = \tau_i(t_i)[\{(\theta_0, t_{-i}) \in \Theta_0 \times T_{-i} : (\theta_0, \theta_{-i}(t_{-i}), \hat{\pi}_i^{k-1}(t_i)) \in E\}].$$

The map  $\hat{\pi}_i: T_i \rightarrow H_{i,\Theta}$ , defined as

$$t_i \mapsto \hat{\pi}_i(t_i) = (\hat{\pi}_i^1(t_i), \hat{\pi}_i^2(t_i), \dots),$$

assigns to each type in a  $\Theta$ -based type space the corresponding  $\Theta$ -hierarchy of beliefs.

From Mertens and Zamir (1985), we know that when  $H_\Theta$  is endowed with the product topology, there is a homeomorphism

$$\phi_i: H_{i,\Theta} \longrightarrow \Delta(\Theta_0 \times \Theta_{-i} \times H_{-i,\Theta})$$

that preserves beliefs of all orders: for each  $\pi_i = (\pi_i^1, \pi_i^2, \dots) \in H_{i,\Theta}$ ,

$$\text{marg}_{\Theta_0 \times \Theta_{-i} \times Z_{-i}^{k-1}} \phi_i(\pi_i) = \pi_i^k \quad \forall k \geq 1.$$

<sup>15</sup>The requirement that  $\theta_i: T_i \rightarrow \Theta_i$  be *onto* is not essential to the results, but is conceptual: if  $\theta_i(T_i) \subsetneq \Theta_i$ , a type space de facto imposes common knowledge (CK) restrictions on payoffs beyond those entailed by the PI structure. Setting  $\Theta_i = \theta_i(T_i)$  guarantees that all CK assumptions on payoffs are represented by the PI structure, so that the type space only imposes extra assumptions on beliefs. This separation is useful in defining the notion of *information invariance* (Section 5). Without the “onto” requirement, that notion would be more involved. (In that case, the notion of embedding, Definition 6, would refer to the images of the  $\theta_i$  functions instead of the sets  $\Theta_i$ .)

Hence, the tuple  $\mathcal{T}_\Theta^* = \langle \Theta, (T_{i,\Theta}^*, \theta_i^*, \tau_i^*)_{i \in N} \rangle$ , where  $T_{i,\Theta}^* := \Theta_i \times H_{i,\Theta}$  and for every  $t_i = (\theta_i, \pi_i) \in T_{i,\Theta}^*$ ,  $\tau_i^*(\theta_i, \pi_i) = \phi_i(\pi_i)$  and  $\theta_i^*(\theta_i, \pi_i) = \theta_i$ , is a type space. It is referred to as the  $(\Theta\text{-based})$  *universal type space*. Let  $\hat{\pi}_i^*(t_i) \equiv \theta_i(t_i)$  and define  $\hat{\pi}_i^*: T_i \rightarrow T_{i,\Theta}^*$  so that  $\hat{\pi}_i^*(t_i) = (\hat{\pi}_i^0(t_i), \hat{\pi}_i(t_i))$ . Mertens and Zamir showed that for any nonredundant type space, the set  $\hat{\pi}^*(T)$  is a *belief-closed* subset of  $T^*$ , in the sense that for every  $\hat{\pi}_i(t_i) \in \hat{\pi}_i(T_i)$ , we have  $\phi_i(\hat{\pi}_i(t_i))[\Theta_0 \times \hat{\pi}_{-i}^*(T_{-i})] = 1$ .<sup>16</sup> A *finite type* is any element  $t_i \in T_{i,\Theta}^*$  that belongs to a finite belief-closed subset of  $T_{i,\Theta}^*$ . The set of finite types is denoted by  $\hat{T}_{i,\Theta}$ .

Players' hierarchies of beliefs (or *types*) are envisioned as purely subjective states describing a player's view of the strategic situation. As such, they enter the analysis as a datum and are regarded in isolation (i.e., player by player and type by type). It is given such (exogenous) beliefs that we can apply game theoretic reasoning to make predictions about players' behavior (the endogenous variables).

#### 2.4. Bayesian Games in Extensive Form

Given a PI structure  $\langle \Theta, (u_i)_{i \in N} \rangle$  and a  $\Theta$ -based type space  $\mathcal{T}$ , let  $\hat{u}_i: \mathcal{Z} \times \Theta_0 \times T \rightarrow \mathbb{R}$  be such that for each  $(z, \theta_0, t) \in \mathcal{Z} \times \Theta_0 \times T$ ,  $\hat{u}_i(z, \theta_0, t) = u_i(z, \theta_0, \theta(t))$ . Function  $\hat{u}_i$  extends  $u_i$ 's domain to the payoff irrelevant higher order beliefs. To avoid unnecessary notation, in the following discussion we use  $u_i$  to denote both payoff functions.

A tuple  $\langle \Theta, \mathcal{T}, (u_i)_{i \in N} \rangle$  and an extensive form  $\Gamma$  define a *Bayesian game in extensive form*:

$$\Gamma^{\mathcal{T}} = \langle N, \mathcal{H}, \mathcal{Z}, \Theta, (T_i, \theta_i, \tau_i, u_i)_{i \in N} \rangle.$$

(Notice that game  $\Gamma^{\mathcal{T}}$  need not be consistent with a common prior.<sup>17</sup>)

### 3. INTERIM SEQUENTIAL RATIONALIZABILITY

Interim sequential rationalizability (ISR) is a solution concept for Bayesian games in extensive form,  $\Gamma^{\mathcal{T}}$ . Similar to rationalizability, ISR is a nonequilibrium solution concept, computed by an iterative deletion procedure. The main difference between ICR and ISR is that the latter is based on *sequential rationality*: at every history, agents play best responses to their conditional conjectures, and the latter are consistent with Bayesian updating whenever possible. This distinction is immaterial in static games, where ISR and ICR coincide.

<sup>16</sup>Type space  $T$  is *nonredundant* if  $\forall t_i, t'_i \in T_i, t_i \neq t'_i$  implies  $\hat{\pi}_i^*(t_i) \neq \hat{\pi}_i^*(t'_i)$ .

<sup>17</sup>Harsanyi's (1967/1968) definition of Bayesian game does not require the existence of a common prior. The common prior assumption corresponds to the special case that Harsanyi called consistent.

*Endogenous Beliefs: Conjectures*

At every history, players hold *conjectures* about their opponents' behavior, their types, and the state of nature. These are represented by conditional probability systems (CPS), that is, arrays of conditional beliefs, one for each history. These beliefs differ from those introduced in Section 2.3 in that they concern and depend on endogenous variables such as the opponents' behavior. Hence, these are *endogenous beliefs*. To avoid confusion, we thus refer to this kind of beliefs as *conjectures*, retaining the term "beliefs" for those introduced in Section 2.3.

For each history  $h \in \mathcal{H}$ , define the event  $[h] \subseteq \Theta_0 \times T_{-i} \times S_{-i}$  as

$$[h] = \Theta_0 \times T_{-i} \times S_{-i}(h).$$

(Notice that, by definition,  $[h] \subseteq [h']$  whenever  $h$  follows  $h'$ .)

DEFINITION 2: A *conjecture* for agent  $i$  is a *conditional probability system* (CPS hereafter), that is a collection  $\mu^i = (\mu^i(h))_{h \in \mathcal{H}}$  of conditional distributions  $\mu^i(h) \in \Delta(\Theta_0 \times T_{-i} \times S_{-i})$  that satisfy the following conditions:

C.1. For all  $h \in \mathcal{H}$ ,  $\mu^i(h) \in \Delta([h])$ .

C.2. For every measurable  $A \subseteq [h] \subseteq [h']$ ,  $\mu^i(h)[A] \cdot \mu^i(h')[h] = \mu^i(h')[A]$ . The set of CPS over  $\Theta_0 \times T_{-i} \times S_{-i}$  is denoted by  $\Delta^{\mathcal{H}}(\Theta_0 \times T_{-i} \times S_{-i})$ .

For each type  $t_i \in T_i$ , his *consistent conjectures* are

$$\Phi_i(t_i) = \left\{ \mu^i \in \Delta^{\mathcal{H}}(\Theta_0 \times T_{-i} \times S_{-i}) : \underset{\Theta_0 \times T_{-i}}{\text{marg}} \mu^i(\phi) = \tau_i(t_i) \right\}.$$

Condition C.1 states that agents are always certain of what they know, that is, the observed public history<sup>18</sup>; condition C.2 states that agents' conjectures are consistent with Bayesian updating whenever possible. Type  $t_i$ 's *consistent conjectures* agree with his beliefs on the environment at the beginning of the game.

*Sequential Rationality*

The set of *sequential best responses* for type  $t_i$  to conjectures  $\mu^i \in \Delta^{\mathcal{H}}(\Theta_0 \times T_{-i} \times S_{-i})$ , denoted by  $r_i(\mu^i|t_i)$ , is defined as

(1)  $s_i \in r_i(\mu^i|t_i)$  if and only if  $\forall h \in \mathcal{H}(s_i)$ ,

$$s_i \in \arg \max_{s_i \in \mathcal{S}_i(h)} \int_{\Theta_0 \times T_{-i} \times S_{-i}} u_i(\mathbf{z}(s_i, s_{-i}), \theta_0, t_{-i}, t_i) d\mu^i(h).$$

<sup>18</sup>Since players are always certain of what they know (see also discussion in Section 2.2), player  $i$ 's conjectures about  $\theta_i$  are omitted: If they were explicitly modelled, condition C.1 would require that conjectures of payoff type  $\theta_i$  satisfy  $\mu^i(h) \in \Delta(\{\theta_i\} \times [h])$  for every  $h \in \mathcal{H}$ .

DEFINITION 3: A strategy  $s_i \in S_i$  is sequentially rational for type  $t_i$ , written  $s_i \in r_i(t_i)$ , if there exists  $\mu^i \in \Phi_i(t_i)$  such that  $s_i \in r_i(\mu^i|t_i)$ .

The notion of sequential rationality is stronger than (normal form) rationality, which only requires that a player optimizes with respect to his initial conjectures  $\mu^i(\phi)$ . The two notions obviously coincide if the game is static (i.e. if  $\mathcal{H} = \{\phi\}$ ).

*Interim Sequential Rationalizability*

Interim sequential rationalizability (ISR) consists of an iterated deletion procedure for each type of each player. The deletion procedure is described as follows: for each type  $t_i$ , reduced strategy  $s_i$  survives the  $k$ th round of deletion if and only if  $s_i$  is sequentially rational for type  $t_i$ , and if it is justified by conjectures  $\mu^i$  that, *at the beginning of the game*, are concentrated on pairs  $(t_{-i}, s_{-i})$  consistent with the previous rounds of deletion.

DEFINITION 4—ISR: For each  $i \in N$ , let  $ISR_i^0 = T_i \times S_i$ . Recursively, for  $k = 1, 2, \dots$  and  $t_i \in T_i$ , let

$$ISR_{-i}^{k-1} = \times_{j \in N \setminus \{i\}} ISR_j^{k-1},$$

$$ISR_i^k(t_i) = \{ \hat{s}_i \in ISR_i^{k-1}(t_i) : \exists \mu^i \in \Phi_i(t_i) \text{ s.t. (i) } \hat{s}_i \in r_i(\mu^i|t_i), \text{ and} \\ \text{(ii) } \text{supp}(\mu^i(\phi)) \subseteq \Theta_0 \times ISR_{-i}^{k-1} \},$$

$$ISR_i^k = \{ (t_i, s_i) \in T_i \times S_i : s_i \in ISR_i^k(t_i) \},$$

and

$$ISR^k = \times_{i \in N} ISR_i^k.$$

Finally,  $ISR := \bigcap_{k \geq 0} ISR^k$ .

Notice that ISR restricts agents' conjectures only at the beginning of the game (condition (ii)). If history  $h$  is given zero probability by the conditional conjectures held at the preceding node,  $i$ 's conjectures at  $h$  may be concentrated anywhere in  $\Theta_0 \times T_{-i} \times S_{-i}(h)$ . ISR therefore corresponds to the assumption that at the beginning of the game, players commonly believe that everyone is (sequentially) rational. But if player  $i$  observes something unexpected, then he may consider that the opponents are not rational. The fundamental logic of ISR is best understood by considering the case of complete information first.

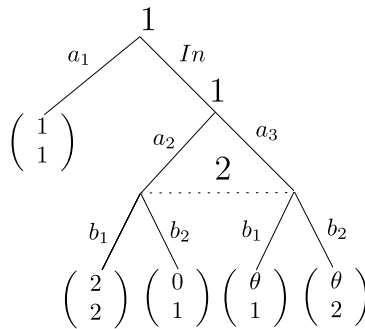


FIGURE 1.—Examples 1 and 2.

EXAMPLE 1: Consider the game in Figure 1, and suppose that it is common knowledge that  $\theta = 0$ . Denote this model by  $T^{\text{CK}} = \{t^{\text{CK}}\}$ . Then strategy  $(\text{In}, a_3)$  is dominated by  $a_1$  and deleted at the first round. Strategies  $a_1$  and  $(\text{In}, a_2)$  are justified by  $b_2$  and  $b_1$ , respectively. Hence,  $\text{ISR}_1^1(t_1^{\text{CK}}) = \{a_1, (\text{In}, a_2)\}$ . Given this, player 2’s initial conjectures in the second round must put zero probability on  $(\text{In}, a_3)$ . No further restrictions are imposed. In particular, conjectures  $\hat{\mu}^2$  can be such that  $\hat{\mu}^2(\phi)[(a_1)] = 1$  and  $\hat{\mu}^2(\text{In})[(\text{In}, a_3)] = 1$ , which makes  $b_2$  the unique sequential best response to  $\hat{\mu}^2$ . On the other hand, if  $\hat{\mu}^2(\phi)[(\text{In}, a_2)] > 0$ , Bayesian updating implies that  $\hat{\mu}^2(\text{In})[(\text{In}, a_2)] = 1$ , and player 2’s unique sequential best response is  $b_1$ . Given that nothing is deleted for player 2, the procedure ends here:  $\text{ISR}(t^{\text{CK}}) = \{a_1, (\text{In}, a_2)\} \times \{b_1, b_2\}$ .

Notice that strategy  $b_2$  (and so  $a_1$ ) is not deleted here because ISR allows player 2 to believe, after an unexpected history, that the opponent may play irrationally (i.e.,  $(\text{In}, a_3)$ ). In contrast, suppose that we make the following assumption:

[H.1] *Even after unexpected moves, player 2 believes that player 1 is rational.*

Then, independent of his initial beliefs, in the proper subgame, player 2 always assigns zero probability to  $(\text{In}, a_3)$  and plays  $b_1$  if rational. Hence, if player 1 believes [H.1] and that player 2 is rational, his unique best response is  $(\text{In}, a_2)$ . This is the logic of Pearce’s (1984) *extensive form rationalizability* (EFR), which delivers  $((\text{In}, a_2), b_1)$  as the unique outcome in this game.

The logic of EFR in Example 1 seems compelling. Yet, as the next example shows, its predictions are not robust. To illustrate the point, we construct a sequence of hierarchies of beliefs, converging to those implicit in Example 1, in which  $(a_1, b_2)$  is the unique ISR outcome (hence, also the unique EFR outcome). Since that outcome is ruled out by EFR in the limit, but uniquely selected along the converging sequence, EFR is not “robust.” Example 2 also provides the intuition for the result in Theorem 1.

EXAMPLE 2: In the game in Figure 1, let the space of uncertainty be such that  $\Theta_1 = \{0, 3\}$ , while  $\Theta_0$  and  $\Theta_2$  are singletons (hence  $\Theta \simeq \Theta_1$ : player 1 is informed, player 2 is not). Let  $t^* = (t_1^*, t_2^*)$  represent the situation in which there is *common certainty* that  $\theta = 0$ :  $t_1^*$  knows that  $\theta = 0$ , and puts probability 1 on  $t_2^*$ ;  $t_2^*$  puts probability 1 on  $\theta = 0$  and  $t_1^*$ . A reasoning similar to that in Example 1 implies that  $\{a_1, (\text{In}, a_2)\}$  and  $\{b_1, b_2\}$  are the sets of ISR strategies for  $t_1^*$  and  $t_2^*$ , written  $\text{ISR}(t^*) = \{a_1, (\text{In}, a_2)\} \times \{b_1, b_2\}$ .

We construct next a sequence of types  $\{t^m\}$ , converging to  $t^*$ , such that  $(a_1, b_2)$  is the unique ISR outcome for each  $t^m$ . Since it is the unique ISR outcome along the sequence, any (strict) refinement that rules it out at  $t^*$  is not robust. (Since  $\mathcal{T}_{\Theta^*}$  is endowed with the product topology, the convergence below is with respect to this topology.)

Fix  $\varepsilon \in (0, \frac{1}{6})$  and let  $p \in (0, \frac{\varepsilon}{(1-2\varepsilon)})$ . Consider the set of type profiles  $T_1^\varepsilon \times T_2^\varepsilon \subseteq \mathcal{T}_{\Theta^*}$ , where  $T_1^\varepsilon = \{-1^3, 1^0, 1^3, 3^0, 3^3, 5^0, 5^3, \dots\}$  and  $T_2^\varepsilon = \{0, 2, 4, \dots\}$ . Types  $k^\theta$  ( $k = -1, 1, 3, \dots, \theta = 0, 3$ ) are player 1's types who know that the true state is  $\theta$ . Beliefs are described as follows. Type  $-1^3$  puts probability 1 on type 0; type 0 assigns probability  $\frac{1}{1+p}$  to type  $-1^3$ , and complementary probability to types  $1^0$  and  $1^3$ , with weights  $(1 - \varepsilon)$  and  $\varepsilon$ , respectively. Similarly, for all  $k = 2, 4, \dots$ , player 2's type  $k$  puts probability  $\frac{1}{1+p}$  on player 1's types  $(k - 1)^0$  and  $(k - 1)^3$ , with weights  $(1 - \varepsilon)$  and  $\varepsilon$ , respectively, and complementary probability  $\frac{p}{1+p}$  on the  $(k + 1)$  types, with weights  $(1 - \varepsilon)$  on  $(k + 1)^0$  and  $\varepsilon$  on  $(k + 1)^3$ . For all other types of player 1, with  $k = 1, 3, \dots$  and  $\theta = 0, 3$ , type  $k^\theta$  puts probability  $\frac{1}{1+p}$  on player 2's type  $k - 1$ , and complementary probability on player 2's type  $k + 1$ . (The type space is represented in Figure 2.) Notice that the increasing sequence of even  $k$ 's and odd  $k$ 's converges to  $t^*$  as we let  $\varepsilon$  approach 0. It will be shown that player 2's types  $0, 2, 4, \dots$  only play  $b_2$ , while player 1's types  $1^0, 3^0, \dots$  only play  $a_1$ .

Clearly, types  $k^3$  ( $k = -1, 1, 3, \dots$ ) play  $(\text{In}, a_3)$ , which they know is dominant. Type 0 puts probability  $\frac{1}{1+p}$  on type  $-1$ , who plays  $(\text{In}, a_3)$ ; given these initial beliefs, type 0's conditional conjectures after In must put probability at

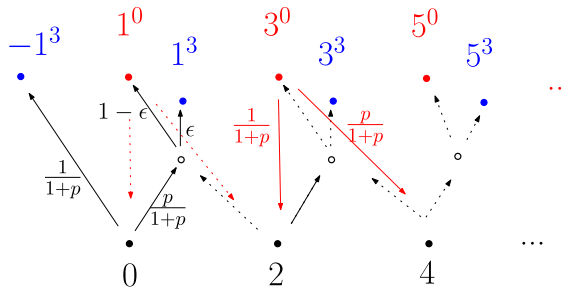


FIGURE 2.—The type space in Example 2.

least  $\frac{1}{1+p}$  on  $(\text{In}, a_3)$ , which makes  $b_2$  optimal. Type  $1^0$  also puts probability  $\frac{1}{1+p}$  on type 0, who plays  $b_2$ ; thus  $a_1$  is the unique best response. Type 2's initial conjectures are such that type  $1^0$  plays  $a_1$ , and types  $1^3$  and  $3^3$  play  $(\text{In}, a_3)$ . Hence, the probability of  $a_3$  conditional on In must be no smaller than

$$\begin{aligned} \Pr(\theta = 3 | \text{not } 1^0) &= \frac{\varepsilon}{1 - \left(\frac{1}{1+p}\right)(1 - \varepsilon)} \\ &= \frac{(1+p)\varepsilon}{p + \varepsilon}. \end{aligned}$$

Given that  $p < \frac{\varepsilon}{(1-2\varepsilon)}$ , this probability is greater than  $\frac{1}{2}$ . Playing  $b_2$  is thus the unique best response, irrespective of type 2's conjectures about the behavior of  $3^0$ . Given this, type  $3^0$  also plays  $a_1$ . The reasoning can be iterated, so that for all types  $1^0, 3^0, 5^0, \dots$ ,  $a_1$  is the unique ISR strategy, while for all types  $0, 2, 4, \dots$  of player 2, strategy  $b_2$  is.

### 3.1. Some Remarks on the Solution Concept

ISR generalizes existing solution concepts that have been defined for special cases. For instance, in games with complete and perfect information, ISR coincides with Ben-El-Mechaieq's (1997) *common certainty in rationality (CCR)*. If the game is static, ISR coincides with ICR. The existing solution concept that is closest in spirit is Battigalli and Siniscalchi's (2007) *weak  $\Delta$ -rationalizability*, which is only defined for games without a type space.<sup>19</sup> Penta (2010b) also showed that, in private values environments, ISR is (generically) equivalent to Dekel and Fudenberg's (1990)  $S^\infty W$  procedure applied to the interim normal form.<sup>20</sup>

Like ICR, one important feature of ISR is that agents' conjectures allow correlation between the opponents' behavior and the state of nature. Correlated conjectures are key to the following results from Penta (2010b), useful for the proof of the structure theorem:

**PROPOSITION 1**—Proposition 1 in Penta (2010b): *ISR is upper hemicontinuous on the universal type space. That is, for each  $t \in T_\theta^*$  and sequence  $\{t^m\}: t^m \rightarrow t$ , and for  $\{s^m\} \subseteq S$  such that  $s^m \rightarrow \hat{s}$  and  $s^m \in \text{ISR}(t^m)$  for every  $m$ ,  $\hat{s} \in \text{ISR}(t)$ .*

<sup>19</sup>I conjecture that extending an analogous result for ICR in Battigalli, Di Tillio, Grillo, and Penta (2011), it can be shown that ISR is equivalent to *weak  $\Delta$ -rationalizability* where the  $\Delta$ -restrictions are those derived from the type space.

<sup>20</sup>The  $S^\infty W$  procedure consists of one round of deletion of weakly dominated strategies followed by iterated deletion of strongly dominated strategies. It was introduced by Dekel and Fudenberg (1990), who showed that  $S^\infty W$  is "robust" to the possibility that players entertain small doubts about their opponents' payoff functions, under the assumption that players *know* their own payoffs, which corresponds to the private values case here.

PROPOSITION 2—Proposition 2 in Penta (2010b): *ISR is type-space-invariant. Let  $\mathcal{T}$  and  $\tilde{\mathcal{T}}$  be two  $\Theta$ -based type spaces. If  $t_i \in T_i, \tilde{t}_i \in \tilde{T}_i$  are such that  $\hat{\pi}^*(t_i) = \hat{\pi}^*(\tilde{t}_i) \in T_{i,\Theta}^*$ , then  $\text{ISR}(t_i) = \text{ISR}(\tilde{t}_i)$ . Indeed, for any  $k$ , if  $t_i$  and  $\tilde{t}_i$  are such that  $\hat{\pi}_i^l(t_i) = \hat{\pi}_i^l(\tilde{t}_i)$  for all  $l \leq k$ , then  $\text{ISR}_i^{k-1}(t_i) = \text{ISR}_i^{k-1}(\tilde{t}_i)$ .*

These results, which can be interpreted as robustness properties (Section 6.1), generalize analogous properties of ICR (Dekel, Fudenberg, and Morris (2007)).

4. THE STRUCTURE THEOREM FOR ISR

The main result in this section shows that, if all common knowledge assumptions on payoffs are relaxed, ISR is the strongest robust solution concept among those that satisfy the minimal requirement discussed in the introduction, ICBSR. (Namely, players are sequentially rational and this is common belief at the beginning of the game.<sup>21</sup>)

Since any PI structure entails common knowledge assumptions, investigating the robustness of solution concepts when no common knowledge assumptions are imposed essentially means to assume that the underlying space of uncertainty  $\Theta$  is sufficiently rich. This is the spirit of Weinstein and Yildiz’s *richness condition*: for each action of each player, there exists a state  $\theta \in \Theta$  in which that action is strictly dominant. This condition cannot be satisfied in dynamic games. An analogous condition though can be formulated simply by adopting a notion of dominance based on sequential rationality.

DEFINITION 5: Strategy  $s_i$  is *conditionally dominant* at  $\theta \in \Theta$  if  $\forall h \in \mathcal{H}(s_i), \forall s'_i \in S_i(h), \forall s_{-i} \in S_{-i}(h)$ ,

$$s_i(h) \neq s'_i(h) \implies u_i(\mathbf{z}(s_i, s_{-i}), \theta) > u_i(\mathbf{z}(s'_i, s_{-i}), \theta).$$

RICHNESS CONDITION (RC): Let  $\langle \Theta_0, (\Theta_i, u_i)_{i \in N} \rangle$  be such that (i)  $\forall s \in S, \exists \theta^s = (\theta_0^s, \theta_i^s, \theta_{-i}^s) \in \Theta : \forall i \in N, s_i$  is *conditionally dominant* at  $\theta^s$  and (ii)  $\forall i \in N, \Theta_i$  is convex.<sup>22</sup>

If the game is static, the notion of conditional dominance coincides with that of strict dominance. RC therefore coincides with Weinstein and Yildiz’s (WY) richness condition in static games. Furthermore, in no information environments (i.e., if  $\Theta \simeq \Theta_0$ ), RC coincides with Chen’s (2011) “extensive form

<sup>21</sup>Penta (2010b) showed that ISR in fact characterizes the behavioral implications of ICBSR.

<sup>22</sup>Condition (ii) in (RC) is a technical assumption, used to perturb payoffs in case of ties between different terminal nodes (see eq. (2), p. 654), which is only required in nongeneric cases. Condition (ii) can be relaxed if, for instance, payoffs are required to be in generic position for every  $\theta \in \Theta$ .



richness.” Hence, the analysis covers both Weinstein and Yildiz’s and Chen’s environments.<sup>23</sup>

**THEOREM 1:** *If  $(\Theta_0, (\Theta_i, u_i)_{i \in N})$  satisfies the richness condition (RC), for any finite type profile  $\hat{t} \in \hat{T}_\Theta$  and any  $s \in \text{ISR}(\hat{t})$ , there exists a sequence of finite type profiles  $\{\hat{t}^m\} \subseteq \hat{T}_\Theta$  such that  $\hat{t}^m \rightarrow \hat{t}$  as  $m \rightarrow \infty$  and  $\text{ISR}(\hat{t}^m) = \{s\}$  for each  $m$ . Furthermore, for each  $m$ ,  $\hat{t}^m$  belongs to a finite belief-closed subset of types,  $T^m \subseteq T_\Theta^*$ , such that for each  $m$  and each  $t \in T^m$ ,  $|\text{ISR}(t)| = 1$ .*

Theorem 1 implies that any refinement of ISR (e.g., extensive form rationalizability or sequential equilibrium) is not upper hemicontinuous (see Example 2 in Section 3). To see this, let  $\mathcal{S}$  be a refinement of ISR (i.e.,  $\mathcal{S}(t) \subseteq \text{ISR}(t)$  for all  $t$ ). Being a refinement, there exist  $t_i \in T_\Theta^*$  and  $s \in \mathcal{S}$  such that  $s \in \text{ISR}(t)$  and  $s \notin \mathcal{S}(t)$ . By Theorem 1, there exists a sequence  $\{t^m\}$  converging to  $t$  such that  $\{s\} = \text{ISR}(t^m) \supseteq \mathcal{S}(t^m)$ , but  $s \notin \mathcal{S}(t)$ . Therefore,  $\mathcal{S}$  is not upper hemicontinuous. Since ISR is upper hemicontinuous (Proposition 1), the following statement is true.

**COROLLARY 1:** *ISR is the strongest upper hemicontinuous solution concept among its refinements.*

The proof of Theorem 1 requires a substantial investment in additional concepts and notation, and is relegated to the Appendix. The main points of departure from Weinstein and Yildiz (2007) are due to the necessity of breaking the ties between strategies at unreached information sets. Although notationally involved, the idea is simple. Consider the sequence constructed in Example 2: to obtain  $b_2$  as the unique ISR strategy for player 2, given that player 1 would play  $a_1$ , it was necessary to perturb player 2’s beliefs so to assign arbitrarily small probability to types  $1^3, 3^3, \dots$ , who believe that  $(\text{In}, a_3)$  is dominant. These “dominance types” play the role of trembles and allow one to break the tie between  $b_1$  and  $b_2$ .

It is proved next that under the richness condition, ISR is generically unique on the universal type space. The proof exploits the following known result.

**LEMMA 1—Mertens and Zamir (1985):** *The set  $\hat{T}_\Theta$  of finite types is dense in  $T_\Theta^*$ , that is,*

$$T_\Theta^* = \text{cl}(\hat{T}_\Theta).$$

<sup>23</sup>Since both Weinstein and Yildiz (2007) and Chen (2011) implicitly assumed  $|\Theta_i| = 1$  for all  $i \in N$ , condition (ii) is vacuously satisfied in their settings. RC thus formally encompasses both settings.

**THEOREM 2:** *Under the richness condition (RC), the set*

$$\mathcal{U} = \{t \in T_{\theta}^* : |\text{ISR}(t)| = 1\}$$

*is open and dense in  $T_{\theta}^*$ . Moreover, the unique ISR outcome is locally constant, in the sense that  $\forall t \in \mathcal{U}$  such that  $\text{ISR}(t) = \{s\}$ , there exists an open neighborhood of types,  $\mathcal{N}_{\delta}(t)$ , such that  $\text{ISR}(t') = \{s\}$  for all  $t' \in \mathcal{N}_{\delta}(t)$ .*

**PROOF:**  *$\mathcal{U}$  is dense.* To show that  $\mathcal{U}$  is dense, notice that by Proposition 2, for any  $\hat{t} \in \hat{T}$  there exists a sequence  $\{\hat{t}^m\} \subseteq \hat{T}^m$  such that  $\hat{t}^m \rightarrow \hat{t}$  and  $\text{ISR}(\hat{t}^m) = \{s\}$  for some  $s \in \text{ISR}(\hat{t})$ . By definition,  $\hat{t}^m \in \mathcal{U}$  for each  $m$ . Hence,  $\hat{t} \in \text{cl}(\mathcal{U})$ , thus  $\hat{T} \subseteq \text{cl}(\mathcal{U})$ . But we know that  $\text{cl}(\hat{T}) = T^*$ ; therefore,  $\text{cl}(\mathcal{U}) \supseteq \text{cl}(\hat{T}) = T^*$ . Hence  $\mathcal{U}$  is dense.

*$\mathcal{U}$  is open and ISR locally constant in  $\mathcal{U}$ .* Since (Proposition 1) ISR is upper hemicontinuous, for each  $t \in \mathcal{U}$  there exists a neighborhood  $\mathcal{N}_{\delta}(t)$  such that for each  $t' \in \mathcal{N}_{\delta}(t)$ ,  $\text{ISR}(t') \subseteq \text{ISR}(t)$ . Since  $\text{ISR}(t) = \{s\}$  for some  $s$  and  $\text{ISR}(t') \neq \emptyset$ , it follows trivially that  $\text{ISR}(t') = \{s\}$ , hence  $\mathcal{N}_{\delta}(t) \subseteq \mathcal{U}$ . Therefore,  $\mathcal{U}$  is open. By the same token, we also have that  $\text{ISR}(t') = \{s\}$  for all  $t' \in \mathcal{N}_{\delta}(t)$ , that is, the unique ISR outcome is locally constant. *Q.E.D.*

**COROLLARY 2:** *Generic uniqueness of ISR implies generic uniqueness of any equilibrium refinement (in particular, of any perfect-Bayesian equilibrium outcome).*

For each  $s \in S$ , let  $\mathcal{U}^s = \{t \in \hat{T}_{\theta} : \text{ISR}(t) = \{s\}\}$ . From Theorem 2, we know that these sets are open. Let the boundary be  $\text{bd}(\mathcal{U}^s) = \text{cl}(\mathcal{U}^s) \setminus \mathcal{U}^s$ .

**COROLLARY 3:** *Under the richness condition, for each  $t \in \hat{T}_{\theta}$ ,  $|\text{ISR}(t)| > 1$  if and only if there exist  $s, s' \in \text{ISR}(t) : s \neq s'$  such that  $t \in \text{bd}(\mathcal{U}^s) \cap \text{bd}(\mathcal{U}^{s'})$ .*

Theorems 1 and 2 together imply a structure theorem for ISR analogous to the one that Weinstein and Yildiz proved, under the no information assumption, for ICR: ISR is a generically unique and locally constant solution concept that yields multiple solutions at, and only at, the boundaries where the concept changes its prescribed behavior.

Since ISR coincides with ICR in static games, Theorems 1 and 2 extend Weinstein and Yildiz’s results to arbitrary information structures.<sup>24</sup>

<sup>24</sup>Given the results above, analogues of the remaining results in Weinstein and Yildiz (2007) can be obtained in a straightforward manner for ISR: in particular, it can be shown that Theorem 1 also holds if one imposes the common prior assumption.

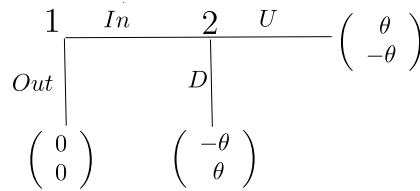


FIGURE 3.—Examples 3 and 4.

#### 4.1. Extensive Form versus Normal Form Approach

Independent work by Chen (2011) considered the robustness question in dynamic games. Chen showed that the structure theorem for ICR holds in dynamic games, once the richness condition is suitably adapted.<sup>25</sup> There are two main differences between Chen's and the approach followed in this paper: first, Chen only considers the case of no information; second, he maintains a normal form approach, that is, he applies ICR to the reduced normal form of the game. These two points are closely related: if no common knowledge restrictions on payoffs are imposed, the no information assumption implies that agents' beliefs on payoffs at unexpected histories are unrestricted. Sequential rationality therefore has no bite at unexpected histories and coincides with normal form rationality.

EXAMPLE 3: Consider the game in Figure 3, and assume that  $\Theta = \{3\}$  (complete information). Denote by  $t^{\text{CK}}$  the degenerate hierarchy in which  $\theta = 3$  is common certainty. Normal form rationalizability (or ICR) does not rule out anything in this game, while ISR delivers the backward induction solution (Out, D).

Now, suppose instead that  $\theta \in \{3, -3\}$  and that agents share common certainty of  $\theta = 3$ , but have no information (hence,  $\Theta^* \simeq \Theta_0^* = \{-3, 3\}$ ). Denote by  $(t_1^A, t_2^A) \in T_{\Theta^*}^*$  players' hierarchies that correspond to (initial) common certainty of  $\theta = 3$ . If player 1 believes that  $\theta = 3$  and that player 2 will play D, player 1's optimal response is to play Out. If player 2 believes this, his information set is unexpected and ISR allows him to revise his beliefs in favor of  $\theta = -3$ . In this case, player 2's best response is U. If player 1 anticipates this, then playing In is optimal even if she believes that  $\theta = 3$ . That is because player 1 knows that player 2, although initially certain of  $\theta = 3$ , does not know that  $\theta = 3$ . Hence,  $\text{ISR}(t^A)$  coincides with  $\text{ICR}(t^A)$  in this example. If players have no information on payoffs, not even one round of backward induction reasoning is robust. In this sense, the very notion of dynamic game is immaterial without information.

<sup>25</sup>Recently, Weinstein and Yildiz (2010) extended Chen's analysis to games with infinite horizon.

This insight has general validity.

**COROLLARY 4:** *If the PI structure has no information and the richness condition is satisfied, ISR and ICR coincide everywhere on the universal type space.*<sup>26</sup>

One possible interpretation of Corollary 4 is that, for what concerns the impact of higher order uncertainty, there is no difference between static and dynamic games. It is important though to emphasize that this is the case under the no information assumption, precisely because the very notion of sequential rationality has no bite in these settings.<sup>27</sup> Under alternative information structures, the normal form approach is inadequate to study the robustness of solution concepts based on sequential rationality (e.g., subgame perfect or sequential equilibrium), which are the main points of interest in dynamic games. There is therefore an intimate connection between agents' information and the extensive form approach adopted here.

### 5. INVARIANCE WITH RESPECT TO INFORMATION

Consider the following example.

**EXAMPLE 4:** As in Example 3, let  $\Theta^* = \{-3, 3\}$ , but this time assume that player 2 observes the realization of  $\theta$  and that this is common knowledge ( $\Theta^* \simeq \Theta_2^* = \{3, -3\}$ ). Consider again the case in which players share initial common certainty of  $\theta = 3$ , denoted by  $t^B = (t_1^B, t_2^B)$ : type  $t_2^B$  knows that  $\theta = 3$  and puts probability 1 on  $t_1^B$ ;  $t_1^B$  puts probability 1 on  $\theta = 3$  and  $t_2^B$ . Since player 1 knows that player 2 knows the true state  $\theta$ , if player 1 is rational and believes that player 2 is rational, he must play Out: if player 2 is rational and knows  $\theta$ , player 1 obtains  $-3$  in the subgame irrespective of the realization of  $\theta$ . The proper subgame at this point is unexpected, but type  $t_2^B$  knows that  $\theta = 3$ , therefore he plays  $D$ . In this case,  $ISR(t^B)$  coincides with the backward induction solution of the complete information model.

Similar to types  $t^A$  from Example 3, types  $t^B$  share the same hierarchy as types  $t^{CK}$ . Yet, while  $ISR(t^A) = ICR(t^{CK}) \supseteq ISR(t^{CK})$ , we have that  $ISR(t^B) = ISR(t^{CK}) \subset ICR(t^{CK})$ . In both examples, the complete information model  $t^{CK}$  was embedded in a richer PI structure,  $\Theta^*$ , and envisioned as the common certainty types  $t^A$  and  $t^B$ , respectively. Only in the second example though could

<sup>26</sup>Theorem 3 in Chen (2011) obtained this result from the structure theorem for ICR and upper hemicontinuity of ISR (Proposition 1). A simple direct proof is also possible, following the argument sketched at the beginning of this section.

<sup>27</sup>In fact, it can be shown that Corollary 4 holds whenever players do not have “enough” information about payoffs. Formally, if for every  $i$  and every  $\theta_i \in \Theta_i, \forall s \in S, \exists(\theta_0^s, \theta_{-i}^s)$  such that  $s_i$  is conditionally dominant at  $(\theta_0^s, \theta_i, \theta_{-i}^s)$ , then ICR and ISR coincide everywhere on the universal type space.

this be done without affecting the predictions of ISR. This suggests a novel notion of invariance.

**DEFINITION 6:** PI structure  $\langle \Theta_0, (\Theta_i, u_i)_{i \in N} \rangle$  is *embedded in* PI structure  $\langle \Theta_0^*, (\Theta_i^*, u_i^*)_{i \in N} \rangle$  if  $\Theta_k \subseteq \Theta_k^*$  for each  $k = 0, 1, \dots, n$  and  $u_i^*(z, \theta) = u_i(z, \theta)$  for all  $(z, \theta, i) \in \mathcal{Z} \times \Theta \times N$ .

In Example 3, the complete information model  $\Theta = \{3\}$  was embedded in the PI structure  $\Theta^* \simeq \Theta_0^* = \{3, -3\}$ , while in Example 4 it was embedded in the PI structure  $\Theta^* \simeq \Theta_2^* = \{3, -3\}$ .

Given a PI structure  $\langle \Theta_0, (\Theta_i, u_i)_{i \in N} \rangle$ , let  $t_i \in T_{i, \Theta}$  denote a type in some  $\Theta$ -based type space. Any such type induces a  $\Theta$ -hierarchy  $\hat{\pi}_i^*(t_i) = (\theta_i(t_i), \hat{\pi}_i^1(t_i), \dots) \in T_{i, \Theta}^*$ . Now consider a richer PI structure  $\langle \Theta_0^*, (\Theta_i^*, u_i^*)_{i \in N} \rangle$  that embeds  $\langle \Theta_0, (\Theta_i, u_i)_{i \in N} \rangle$ . Since  $\Theta_k \subseteq \Theta_k^*$  for all  $k$ ,  $\hat{\pi}_i^*(t_i)$  can be naturally embedded in the  $\Theta^*$ -based universal type space and can be seen as a  $\Theta^*$ -based hierarchy. Let  $\beta_i: T_{i, \Theta}^* \rightarrow T_{i, \Theta^*}^*$  denote such embedding and let  $\kappa_i^* \equiv \beta_i \circ \hat{\pi}_i^*$ .

In Examples 3 and 4, the common certainty types  $t^A$  and  $t^B$  have the same hierarchy as the common knowledge type  $t^{\text{CK}}$  (e.g.,  $\kappa_i^*(t_i^{\text{CK}}) = t_i^A \in T_{i, \Theta^*}^*$ ).

**DEFINITION 7:** A solution concept  $\mathcal{S}_i: T_i \rightrightarrows S_i$  is *information-invariant* if, for any  $\langle \Theta_0, (\Theta_i, u_i)_{i \in N} \rangle$  and  $\langle \Theta'_0, (\Theta'_i, u'_i)_{i \in N} \rangle$  embedded in  $\langle \Theta_0^*, (\Theta_i^*, u_i^*)_{i \in N} \rangle$ , and for any  $t_i \in T_{i, \Theta}$  and  $t'_i \in T_{i, \Theta'}$ , if  $\kappa_i^*(t_i) = \kappa_i^*(t'_i)$ , then  $\mathcal{S}_i(t_i) = \mathcal{S}_i(t'_i)$ .

As shown by Example 3, ISR is not information-invariant in general, while it satisfies information invariance in Example 4. The reason for such different behavior is that under the no information assumption, moving to a PI structure with a richer set of states entails more freedom to specify a player's beliefs about his own payoffs, thereby changing the set of sequential best responses. In contrast, in Example 4, player 2 knows his own payoffs (and this is common knowledge). Hence, even if ISR leaves players' beliefs at unexpected histories unrestricted, a type's preferences over the terminal nodes do not change. This provides the intuition for the information-invariance result in private values settings.

**PROPOSITION 3:** *In static games, ISR is information-invariant. In dynamic games, ISR is information-invariant in environments with private values.*

For the proof, see Appendix B.

The first part of Proposition 3 follows trivially from the fact that the assumptions on information only play a role at unexpected histories. The driving force behind the second part of the proposition is that in environments with private values, players know their payoffs and therefore their beliefs about them never

change.<sup>28</sup> It is only such belief stability that really matters for the invariance result. (See the discussion in Section 6.2.)

It is easy to see that information invariance holds also outside of private values (e.g., Example 4 is not with private values). For instance, it can be shown that Proposition 3 holds provided that players have enough information on their preference ranking over the terminal histories. That is, if the PI structures are such that for every  $i$  and every  $\theta_i \in \Theta_i$ , for any  $(\theta_0, \theta_{-i}), (\theta'_0, \theta'_{-i}) \in \Theta_0 \times \Theta_{-i}$  and any  $z, z' \in \mathcal{Z}$ ,  $u_i(z, \theta_i, (\theta_0, \theta_{-i})) > u_i(z', \theta_i, (\theta_0, \theta_{-i}))$  if and only if  $u_i(z, \theta_i, (\theta'_0, \theta'_{-i})) > u_i(z', \theta_i, (\theta'_0, \theta'_{-i}))$ .

## 6. DISCUSSION

### 6.1. Robustness(-es)

If we let  $\Theta^*$  denote a space of uncertainty that satisfies the richness condition and we let  $\mathcal{T}_{\Theta^*}$  denote the  $\Theta^*$ -based universal type space, assuming common knowledge of  $\mathcal{T}_{\Theta^*}$  entails no essential loss of generality.  $\mathcal{T}_{\Theta^*}$  can therefore be thought of as a *universal model* and can be used to investigate the robustness of game theoretic predictions “when *all* common knowledge assumptions are relaxed.” The continuity of a solution concept on such a universal model therefore corresponds to a specific robustness property: robustness with respect to small “mistakes” in the modelling choice of which subset of players’ hierarchies to consider. Proposition 1 states that ISR is robust in this sense, under all information structures. Furthermore, Theorem 1 implies that no refinement of ISR is robust in this sense.

In modelling a strategic situation, when a subset of  $\Theta^*$ -hierarchies is selected, it is common to represent them by means of (nonuniversal)  $\Theta^*$ -based type spaces,  $\mathcal{T}_{\Theta^*}$  (Definition 1). This modelling practice does not change the common knowledge assumptions on the PI structure, but imposes restrictions on beliefs that entail some loss in generality. A solution concept is *type-space-invariant* if the behavior prescribed for a given hierarchy does not depend on whether it is represented as an element of  $\mathcal{T}_{\Theta^*}$  or as a type in a (nonuniversal)  $\Theta^*$ -based type space. Proposition 2 therefore states another robustness property for ISR. This one also holds irrespective of the information structure.

We typically make common knowledge assumptions not only on players’ beliefs, but also on payoffs. For instance, if all types in  $\mathcal{T}_{\Theta^*}$  have beliefs concentrated on some strict subset  $\Theta \subset \Theta^*$ , that is, there is common certainty of  $\Theta$  in  $\mathcal{T}_{\Theta^*}$ , it is common to exclude from the model states in  $\Theta^* \setminus \Theta$ . This way, *common certainty* of  $\Theta$  is turned into *common knowledge* of  $\Theta$ , and hierarchies are represented by  $\Theta$ -based type spaces, where  $\Theta$  is a PI structure embedded in  $\Theta^*$

<sup>28</sup>Such “beliefs stability” is implicit in our setup, because  $i$ ’s beliefs about  $\theta_i$  are not modelled explicitly. But, as discussed in Section 2.2, explicitly modelling such beliefs would be redundant, because knowledge implies beliefs stability.

(Definition 6). A solution concept is *information-invariant* (Definition 7) if its predictions are robust to the introduction of these extra common knowledge assumptions. Information invariance is therefore yet another robustness property, not previously considered by the literature.<sup>29</sup> As discussed in Section 5, ISR is *information-invariant* in environments with private values, not in environments with no information.

### 6.2. A Subjectivist View: Strong Belief

The setting of this paper rests on two assumptions:

A.1. Every player *knows* his own payoff type in every state.

A.2. A player with payoff types  $\hat{\theta}_i$  never abandons the belief that  $\theta_i = \hat{\theta}_i$  (while he may abandon beliefs about  $(\theta_0, \theta_{-i})$ ).

As discussed in Section 2.2, within the classical “possible worlds” models, condition A.2 is a natural consequence of the requirement (implicit in standard models in information economics) that agents’ beliefs do not contradict their information.

An alternative interpretation of payoff states is also possible and does not require the notion of an external true state. Under this interpretation, A.2 does not follow naturally from A.1, and it may be interesting to look at the two assumptions separately. In this alternative “subjectivist” view, payoff types  $\hat{\theta}_i$  represent those beliefs about payoffs that player  $i$  is never willing to abandon (his *strong beliefs*). A payoff state no longer represents information about the true payoffs anymore, but a realization of individuals’ epistemic states. None of the results above is affected by this alternative interpretation. I maintain the terminology of the “possible worlds” model simply because it is the standard paradigm in information economics.<sup>30</sup>

From this perspective, it is easy to see how a proper relabelling of the model can accommodate several variations of A.2. For instance, relaxing A.2 in a given environment is always equivalent to considering some other environment with no information. Consider the following example: Let  $N = \{1, 2\}$ ,  $\Theta_0 := \Theta_0^1 \times \Theta_0^2 = [1, 2]^2$ ,  $\Theta_i = [0, 3]$ , and  $u_i(z, \theta) = (\theta_i - \theta_0^i)U_i(z)$  for some  $U_i(\cdot) \in [0, 1]^z$ . Player  $i$  observes  $\theta_i$ , but not  $\theta_0^i$ . To relax A.2, we can reformulate the model, letting  $\hat{\Theta}_0 = \hat{\Theta}_0^1 \times \hat{\Theta}_0^2 = [-2, 2]^2$  and  $\hat{u}_i(z, \theta) = \hat{\theta}_0^i \cdot U_i(z)$  for each  $i = 1, 2$ . Similar transformations can also accommodate some strengthened versions of A.2, in which players entertain strong beliefs about components other than  $\theta_i$ . Suppose, for instance, that player  $i$  is never willing to change his beliefs about  $\theta_0^i$ . Then the environment can be rewritten as one

<sup>29</sup>Dekel, Fudenberg, and Morris (2007) studied the upper hemicontinuity and type space invariance of ICR.

<sup>30</sup>The subjectivist view is well known in other fields such as philosophy and computer science. For instance, the classic textbook by Gardenfors (1988) analyzes the logic of epistemic change without any reference to an external state and the corresponding notions of truth and falsehood.

of private values, setting  $\hat{\Theta}_i = [-2, 2]$  and  $\hat{u}_i(z, \theta) = \hat{\theta}_i \cdot U_i(z)$  for each  $i = 1, 2$ . Due to the restrictiveness of the information partition considered here (with a product structure), not all strong beliefs can be accommodated this way. (For instance, if  $\theta_0^1$  and  $\theta_0^2$  in the example above are replaced by a common value component  $\theta_0$ , then the product structure cannot accommodate strong belief in  $\theta_0$ .) However, a similar relabelling is always possible if general information partitions over  $\Theta$  are considered. This generalization seems to be conceptually straightforward, so I leave it to future research.

APPENDIX

A. Proving Theorem 1

The proof exploits a refinement of ISR—strict sequential rationalizability (SSR)—in which strategies that are never *strict* sequential best responses are deleted at each round. The proof involves two main steps: in the first step (Lemma 3), it is shown that if  $s_i \in \text{ISR}_i(t_i)$  for finite type  $t_i$ , then  $s_i$  is also SSR for some nearby type  $t'_i$ ; in the second step (Lemma 4), it is shown that by perturbing beliefs further, any  $s_i \in \text{SSR}_i(t'_i)$  can be made uniquely ISR for a type close to  $t'_i$ .

DEFINITION 8: Fix a  $\Theta$ -based type space,  $\mathcal{T} = \langle \Theta_0, (\Theta_i, T_i, \theta_i, \tau_i)_{i \in N} \rangle$ . Let  $\text{SSR}_i^0 = T_i \times S_i$ . Recursively, for each  $k = 1, 2, \dots$  and  $t_i \in T_i$ , let

$$\text{SSR}_{-i}^{k-1} = \times_{j \neq i, 0} \text{SSR}_j^{k-1},$$

$$\text{SSR}_i^k(t_i) = \left\{ \hat{s}_i \in \text{SSR}_i^{k-1}(t_i) : \exists \mu^i \in \Phi_i(t_i) \text{ s.t.} \right.$$

$$\text{(i) } r_i(\mu^i | t_i) = \{\hat{s}_i\} \text{ (ii) } \text{supp}(\mu^i(\phi)) \subseteq \Theta_0^* \times \text{SSR}_{-i}^{k-1}$$

$$\text{(iii) if } t_{-i} \in \text{supp}\left(\text{marg}_{T_{-i}, \Theta} \mu^i(\phi)\right) \text{ and } s_{-i} \in \text{SSR}_{-i}^{k-1}(t_{-i}),$$

$$\text{then: } s_{-i} \in \text{supp}\left(\text{marg}_{S_{-i}} \mu^i(\phi)\right) \left. \right\},$$

$$\text{SSR}_i^k = \{(t_i, s_i) \in T_i \times S_i : s_i \in \text{SSR}_i^k(t_i)\},$$

and

$$\text{SSR}^k = \times_{i \in N} \text{SSR}_i^k.$$

$$\text{Finally, } \text{SSR} = \bigcap_{k \geq 0} \text{SSR}^k.$$

The following lemma states a standard fixed-point property for SSR.



LEMMA 2: Let  $\{V_j\}_{j \in N}$  be such that for each  $i \in N$ ,  $V_i \subseteq S_i \times T_i$ , and  $\forall s_i \in V_i(t_i)$ ,  $\exists \mu^i \in \Phi_i(t_i)$ ,

(i)  $\text{supp}(\mu^i(\phi)) \subseteq \times_{j \neq i} V_j$ ,

(ii)  $\{s_i\} = r_i(\mu^i|t_i)$ .

Then  $V_i(t_i) \subseteq \text{SSR}_i(t_i)$ .

Exploiting the richness condition, let  $\bar{\Theta} \subset \Theta$  be a finite set of dominance states, such that  $\forall s \in S$ ,  $\exists! \theta^s \in \bar{\Theta}$  at which  $s$  is conditionally dominant. For each  $s \in S$ , let  $\bar{t}^s \in T_{\bar{\Theta}}^*$  be such that for each  $i$ ,  $\theta_i(\bar{t}^s) = \theta_i^s$  and  $\tau_i(\bar{t}_i^s)[\theta_0^s, \bar{t}_{-i}^s] = 1$ . Let  $\bar{T} = \{\bar{t}^s : s \in S\}$ , and let  $\bar{T}_j$  and  $\bar{T}_{-j}$  denote the corresponding projections. Elements of  $\bar{T}_i$  will be referred to as *dominance types*, and will play the role of the  $k^{a_3}$  types in Example 2.

For each  $i$  and  $s_i \in S_i$ , let  $\bar{T}_{-i}(s_i)$  be such that  $\forall s_{-i} \in S_{-i}$ ,  $\exists! \bar{t}_{-i} \in \bar{T}_{-i}(s_i)$  such that  $\bar{t}_{-i} = \bar{t}_{-i}^{(s_i, s_{-i})}$ . Notice that for each  $\bar{t}_i^s \in \bar{T}_i$ ,  $\{s_i\} = \text{SSR}_i^1(\bar{t}_i^s)$ , because  $s_i$  is the unique sequential best reply to any conjecture consistent with condition (iii) in Definition 8.

LEMMA 3: Under the richness condition, for any finite type  $t_i \in \hat{T}_{i,\theta}$ , for any  $s_i \in \text{ISR}_i(t_i)$ , there exists a sequence of finite types  $\{t_i^m, s_i^m\}_{m \in \mathbb{N}}$ , such that the following statements hold:

(i)  $t_i^m, s_i^m \rightarrow t_i, s_i$  as  $m \rightarrow \infty$ .

(ii)  $\forall m, s_i^m \in \text{SSR}_i(t_i^m, s_i^m)$  and  $t_i^m, s_i^m \in \hat{T}_{i,\theta}$ .

(iii)  $\forall m$ , conjectures  $\mu^{s_i^m, m} \in \Phi(t_i^m, s_i^m)$  such that  $\{s_i^m\} = r_i(\mu^{s_i^m, m} | t_i^m, s_i^m)$  satisfy

$$\bar{T}_{-i}(s_i) \subseteq \text{supp}\left(\text{marg}_{T_{-i,\theta}^*} \mu^{s_i^m, m}(\phi)\right).$$

PROOF: Step I: Fix  $t_i \in \hat{T}_{i,\theta}$ . For each  $k \neq i$ , let  $\Theta_k$  be the finite set of payoff states that receive positive probability by  $t_i$ . For each  $s_i \in \text{ISR}(t_i)$ ,  $\exists \mu^{s_i} \in \Phi_i(t_i)$  such that (i)  $s_i \in r_i(\mu^{s_i} | t_i)$  and (ii)  $\text{supp}(\mu^{s_i}(\phi)) \subseteq \Theta_0 \times \text{ISR}_{-i}$ . Given a probability space  $(\Omega, \mathcal{B})$  and a set  $A \in \mathcal{B}$ , denote by  $v_{[A]}$  the uniform probability distribution concentrated on  $A$ . For each  $\varepsilon \in [0, 1]$ , consider the set of types profiles  $\times_{i \in N} T_i^\varepsilon \subseteq T_{\bar{\Theta}}^*$  such that each  $T_i^\varepsilon$  consists of all the types  $\bar{t}_i \in \bar{T}_i$  and of types  $\bar{t}_i(t_i, s_i, \varepsilon)$  such that

$$(2) \quad \theta_i(\bar{t}_i(t_i, s_i, \varepsilon)) = \varepsilon \theta_i^s + (1 - \varepsilon) \theta_i(t_i)$$

and

$$(3) \quad \tau_i^\varepsilon(\bar{t}_i(t_i, s_i, \varepsilon)) = \varepsilon v_{\{[\theta_0^s] \times \bar{T}_{-i}(s_i)\}} + (1 - \varepsilon)[\mu^{s_i}(\phi) \circ \hat{t}_{-i,\varepsilon}^{-1}],$$

where  $\bar{T}_{-i} \subseteq T_{-i}^\varepsilon$  is the subset of *dominance type* profiles defined above, and

$$\hat{t}_{-i,\varepsilon} : \Theta_0 \times T_{-i} \times S_{-i} \rightarrow \Theta_0 \times T_{-i}^\varepsilon$$

is such that

$$\hat{\tau}_{-i,\varepsilon}(\theta_0, s_{-i}, t_{-i}) = (\theta_0, \bar{\tau}_{-i}(t_{-i}, s_{-i}, \varepsilon)).$$

By construction, with probability  $\varepsilon$ , type  $\bar{\tau}_{-i}(t_i, s_i, \varepsilon)$  is certain that  $s_i$  is conditionally dominant and puts positive probability on all of the opponents' dominance types in  $\bar{T}_{-i}$ . Define  $\gamma: \Theta_0 \times T_{-i}^\varepsilon \rightarrow \Theta_0 \times T_{-i}^\varepsilon \times S_{-i}$  such that

$$\begin{aligned} &\text{for every } \bar{\tau}_{-i}(t_{-i}, s_{-i}, \varepsilon) \in T_{-i}^\varepsilon \setminus \bar{T}_{-i}, \\ &\gamma(\theta_0, \bar{\tau}_{-i}(t_{-i}, s_{-i}, \varepsilon)) = (\theta_0, \bar{\tau}_{-i}(t_{-i}, s_{-i}, \varepsilon), s_{-i}) \end{aligned}$$

and for every

$$\bar{\tau}_{-i}^s \in \bar{T}_{-i} \subseteq T_{-i}^\varepsilon, \quad \gamma(\theta_0, \bar{\tau}_{-i}^s) = (\theta_0, \bar{\tau}_{-i}^s, s_{-i}).$$

Consider the conjectures  $\hat{\mu}^i \in \Delta^{\mathcal{H}}(\Theta_0 \times T_{-i}^\varepsilon \times S_{-i})$  defined by

$$\hat{\mu}^i(\phi) = (\tau_i^\varepsilon(\bar{\tau}_i(t_i, s_i, \varepsilon)) \circ \gamma^{-1}) \in \Delta(\Theta_0 \times T_{-i}^\varepsilon \times S_{-i}).$$

For any  $\varepsilon > 0$ , the conjectures  $\hat{\mu}^i$  are such that  $\bar{T}_{-i} \subseteq \text{supp}(\text{marg}_{T_{-i}^\varepsilon} \hat{\mu}^i(\phi))$ . From the definition of  $\gamma$ , it follows that  $\text{supp}(\text{marg}_{S_{-i}} \hat{\mu}^i(\phi)) = S_{-i}$ , so that the entire CPS  $(\hat{\mu}^i(h))_{h \in \mathcal{H}}$  can be obtained via Bayes' rule. This also implies that  $\hat{\mu}^i$  satisfies condition (iii) in the definition of SSR. Furthermore, by construction,  $\hat{\mu}^i \in \Phi_i(\bar{\tau}_i(t_i, s_i, \varepsilon))$ , and  $\forall \varepsilon > 0, \forall h \in \mathcal{H}, \exists \eta^{\varepsilon,h} \in (0, 1)$  such that  $\eta^{\varepsilon,h} \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and

$$\begin{aligned} \text{marg}_{\Theta_0 \times \Theta_{-i} \times S_{-i}} \hat{\mu}^i(h) &= \eta^{\varepsilon,h} \cdot \text{marg}_{\Theta_0 \times \Theta_{-i} \times S_{-i}} (v_{\{\{\theta_0^s\} \times \bar{T}_{-i}^h(s_i) \times S_{-i}\}}) \\ &\quad + (1 - \eta^{\varepsilon,h}) \cdot \text{marg}_{\Theta_0 \times \Theta_{-i} \times S_{-i}} \mu^{s_i}(h), \end{aligned}$$

where  $\bar{T}_{-i}^h(s_i) = \{\bar{\tau}_{-i}^{(s_i, s_{-i})} : s_{-i} \in S_{-i}(h)\}$ . Hence, the conditional conjectures  $\hat{\mu}^i(h)$  of type  $\bar{\tau}_{-i}(t_{-i}, s_{-i}, \varepsilon)$  are a mixture: with probability  $(1 - \eta^{\varepsilon,h})$ , they agree with  $\mu^{s_i}(\phi)$ , which made  $s_i$  the sequential best response; with probability  $\eta^{\varepsilon,h}$ , they are concentrated on payoff states  $\{\theta_0^s\} \times \{\theta_{-i}(t_{-i}) : t_{-i} \in \bar{T}_{-i}^h(s_i)\}$ , which together with the fact that  $\theta_i(\bar{\tau}_i(t_i, s_i, \varepsilon)) = \varepsilon \theta_i^{s_i} + (1 - \varepsilon) \theta_i(t_i)$  breaks all the ties in favor of  $s_i$ , so that  $r_i(\hat{\mu}^i | \bar{\tau}_i(t_i, s_i, \varepsilon)) = \{s_i\}$ . Thus,  $s_i \in \text{SSR}_i(\bar{\tau}_i(t_i, s_i, \varepsilon))$ , so that (ii) and (iii) in the lemma are satisfied for all  $\varepsilon > 0$ .

The remainder of the proof guarantees that part (i) in the lemma also holds and it is identical to WY's counterpart.

*Step II.* We show that  $\hat{\pi}_i^*(\bar{\tau}_i(t_i, s_i, \varepsilon)) \rightarrow \hat{\pi}_i^*(t_i)$  as  $\varepsilon \rightarrow 0$ . By construction,  $\tau_i(\bar{\tau}_i(t_i, s_i, \varepsilon))$  are continuous in  $\varepsilon$ , hence  $\hat{\pi}_i^*(\bar{\tau}_i(t_i, s_i, \varepsilon)) \rightarrow \hat{\pi}_i^*(\bar{\tau}_i(t_i, s_i, 0))$  as  $\varepsilon \rightarrow 0$ . It suffices to show that  $\hat{\pi}_i^*(\bar{\tau}_i(t_i, s_i, 0)) = \hat{\pi}_i^*(t_i)$  for each  $t_i$  and  $i$ . This is proved by induction. The payoff types and the first-order beliefs are the same.

For the inductive step, assume that  $(\hat{\pi}_i^l(\bar{t}_i(t_i, s_i, 0)))_{l=0}^{k-1} = (\hat{\pi}_i^l(t_i))_{l=0}^{k-1}$ . We show that  $\hat{\pi}_i^k(\bar{t}_i(t_i, s_i, 0)) = \hat{\pi}_i^k(t_i)$ . Define  $D_{-i}^{k-1} = \{(\hat{\pi}_{-i}^l(t_{-i}))_{l=0}^{k-1} : t_{-i} \in T_{-i}\}$ . Under the inductive hypothesis, it can be shown (see WY) that

$$(4) \quad \text{marg}_{\theta_0 \times D_{-i}^{k-1}} [\mu^i(\phi) \circ \hat{t}_{-i}^{-1}] = \text{marg}_{\theta_0 \times D_{-i}^{k-1}} \mu^i(\phi).$$

Therefore,

$$\begin{aligned} \hat{\pi}_i^k(\bar{t}_i(t_i, s_i, 0)) &= \mathbf{v}_{[\hat{\pi}_{-i}^{k-1}(\bar{t}_i(t_i, s_i, 0))]} \times \text{marg}_{\theta_0 \times D_{-i}^{k-1}} [\mu^{s_i}(\phi) \circ \hat{t}_{-i}^{-1}] \\ &= \mathbf{v}_{[\hat{\pi}_{-i}^{k-1}(\bar{t}_i(t_i, s_i, 0))]} \times \text{marg}_{\theta_0 \times D_{-i}^{k-1}} \mu^{s_i}(\phi) \\ &= \mathbf{v}_{[\hat{\pi}_{-i}^{k-1}(t_{-i})]} \times \text{marg}_{\theta_0 \times D_{-i}^{k-1}} \mu^{s_i}(\phi) \\ &= \mathbf{v}_{[\hat{\pi}_{-i}^{k-1}(t_{-i})]} \times \text{marg}_{\theta_0 \times D_{-i}^{k-1}} \tau_i(t_i) = \hat{\pi}_i^k(t_i), \end{aligned}$$

where the first equality is the definition of  $k$ th level belief, the second equality is from (4), the third equality is from the inductive hypothesis, the fourth inequality is from the fact that  $\mu^{s_i} \in \Phi(t_i)$ , and the last inequality is again by definition. *Q.E.D.*

LEMMA 4: *Under the richness condition, for each finite type  $\hat{t}_i \in \hat{T}_{i,\theta}$ , for each  $k$ , for each  $s_i \in \text{SSR}_i^k(\hat{t}_i)$  such that the conjectures  $\mu^{s_i} \in \Phi(\hat{t}_i) : \{s_i\} = r_i(\mu^{s_i} | \hat{t}_i)$  satisfy*

$$\tilde{T}_{-i}(s_i) \subseteq \text{supp}_{T_{-i,\theta}^*}(\text{marg } \mu^{s_i}(\phi)),$$

there exists  $\tilde{t}_i \in \hat{T}_i$  such that the following statements hold:

- (i) For each  $k' \leq k$ ,  $\hat{\pi}^{k'}(\tilde{t}_i) = \hat{\pi}^{k'}(\hat{t}_i)$ .
- (ii)  $\text{ISR}_i^{k+1}(\tilde{t}_i) = \{s_i\}$ .
- (iii)  $\tilde{t}_i \in \tilde{T}_i^{\tilde{t}_i}$  for some finite belief-closed set of types  $\tilde{T}_i^{\tilde{t}_i} = \times_{j \in N} \tilde{T}_j^{\tilde{t}_i}$  such that  $|\text{ISR}^{k+1}(t)| = 1$  for each  $t \in \tilde{T}_i^{\tilde{t}_i}$ .

Hence, for any such  $s_i \in \text{SSR}_i(\hat{t}_i)$  there exists a sequence of finite types  $t_{i,m} \rightarrow \hat{t}_i$  such that  $\text{ISR}_i(t_{i,m}) = \{s_i\}$ .

PROOF: The proof is by induction. For  $k = 0$ , let  $\tilde{t}_i$  be such that  $\theta_i(\tilde{t}_i) = \theta_i(\hat{t}_i)$  and  $\tau_i(\tilde{t}_i) = \mathbf{v}_{\{[\theta^s] \times \tilde{T}_{-i}(s_i)\}}$ . Clearly,  $\text{ISR}_i^1(\tilde{t}_i) = \{s_i\}$  and condition (i) is satisfied. Fix  $k > 0$ , and write each  $t_{-i} = (\lambda, \kappa)$ , where  $\lambda = \{\hat{\pi}_{-i}^{k'}(t_{-i})\}_{k'=0}^{k-1}$  and  $\kappa = \{\hat{\pi}_{-i}^{k'}(t_{-i})\}_{k'=k}^\infty$ . Let  $L_{-i}^{k-1} = \{\lambda : \exists \kappa \text{ s.t. } (\lambda, \kappa) \in T_{-i}^*\}$ . As the inductive hypothesis,

assume that for each finite  $t_{-i} = (\lambda, \kappa)$  and  $s_{-i} \in \text{SSR}_{-i}^{k-1}(t_{-i})$  such that  $\bar{T}_{-i}(s_i) \subseteq \text{supp}(\text{marg}_{\hat{T}_{-i}} \mu^{s_i}(\phi))$ , there exists finite  $t_{-i}^{s_{-i}} = (\lambda, \kappa^{s_{-i}, \lambda})$  such that  $\text{ISR}_i^k(t_i^{s_i}) = \{s_i\}$ . Taking any  $s_i \in \text{SSR}_i^k(\hat{t}_i)$  such that  $\bar{T}_{-i}(s_i) \subseteq \text{supp}(\text{marg}_{\hat{T}_{-i}} \mu^{s_i}(\phi))$ , we construct a type  $\tilde{t}_i$  such that for each  $k' \leq k$ ,  $\hat{\pi}_i^{k'}(\hat{t}_i) = \hat{\pi}_i^{k'}(\tilde{t}_i)$  and  $\text{ISR}_i^{k+1}(\tilde{t}_i) = \{s_i\}$ . By definition, if  $s_i \in \text{SSR}_i^k(\hat{t}_i)$ ,  $\exists \mu^{s_i} \in \Delta^{\mathcal{H}}(\Theta_0 \times T_{-i, \Theta}^* \times S_{-i})$  such that

$$\begin{aligned} \tau_i(\hat{t}_i) &= \text{marg}_{\Theta \times T_{-i}} \mu^{s_i}(\phi), \\ \text{supp}(\mu^{s_i}(\phi)) &\subseteq \Theta_0 \times \text{SSR}_{-i}^{k-1}, \\ \{s_i\} &= r_i(\mu^{s_i}|\hat{t}_i). \end{aligned}$$

Using the inductive hypothesis, define the mapping

$$\begin{aligned} \varphi : \bigcup_{h \in \mathcal{H}} \left[ \text{supp} \left( \text{marg}_{\Theta_0 \times L_{-i}^{k-1} \times S_{-i}} \mu^{s_i}(h) \right) \right] &\rightarrow \Theta_0 \times T_{-i, \Theta}^* \\ \text{s.t. } \varphi(\theta, \lambda, s_{-i}) &= (\theta, (\lambda, \kappa^{s_{-i}, \lambda})). \end{aligned}$$

Define type  $\tilde{t}_i$  as

$$\begin{aligned} \tau_i(\tilde{t}_i) &= \text{marg}_{\Theta_0 \times L_{-i}^{k-1} \times S_{-i}} \mu^{s_i}(\cdot|\phi) \circ \varphi^{-1} \\ &= \mu^{s_i}(\phi) \circ \text{proj}_{\Theta_0 \times L_{-i}^{k-1} \times S_{-i}}^{-1} \circ \varphi^{-1}. \end{aligned}$$

By construction (for the inductive hypothesis), the first  $k$  orders of beliefs are the same for  $t_i$  and  $\tilde{t}_i$  (which is point (i) in the lemma):

$$\begin{aligned} \tilde{\pi}_i^k(\tilde{t}_i) &= \text{marg}_{\Theta_0 \times L_{-i}^{k-1}} \tau_i(\tilde{t}_i) \\ &= \mu^{s_i}(\phi) \circ \text{proj}_{\Theta_0 \times L_{-i}^{k-1} \times S_{-i}}^{-1} \circ \varphi^{-1} \circ \text{proj}_{\Theta_0 \times L_{-i}^{k-1}}^{-1} \\ &= \mu^{s_i}(\phi) \circ \text{proj}_{\Theta_0 \times L_{-i}^{k-1}}^{-1} \\ &= (\mu^{s_i}(\phi) \circ \text{proj}_{\Theta_0 \times T_{-i, \Theta}^*}^{-1}) \circ \text{proj}_{\Theta_0 \times L_{-i}^{k-1}}^{-1} \\ &= \text{marg}_{\Theta_0 \times L_{-i}^{k-1}} \tau_i(t_i) = \tilde{\pi}_i^k(t_i). \end{aligned}$$

The first equality is by definition, the second equality is from construction of  $\tau_i(\tilde{t}_i)$ , and the third equality is from the definition of  $\varphi$ , for which

$$\text{proj}_{\Theta_0 \times L_{-i}^{k-1} \times S_{-i}}^{-1} \circ \varphi^{-1} \circ \text{proj}_{\Theta_0 \times L_{-i}^{k-1}}^{-1} = \text{proj}_{\Theta_0 \times L_{-i}^{k-1}}^{-1}.$$

The fourth and fifth equalities are notational, and the last equality is by definition. We need to show that  $\text{ISR}_i^{k+1}(\tilde{t}_i) = \{s_i\}$ . To this end, notice that each  $(\theta_0, t_{-i}) \in \text{supp}(\tau_i(\tilde{t}_i))$  is of the form  $(\theta_0, t_{-i}) = (\theta_0, (\lambda, \kappa^{s_{-i}, \lambda}))$  and it is such that  $\text{ISR}_{-i}^k((\lambda, \kappa^{s_{-i}, \lambda})) = \{s_{-i}\}$ . Hence, the CPS consistent with  $\tilde{t}_i$  and with the restrictions of  $\text{ISR}_{-i}^k$  are  $\tilde{\mu}^i$  such that

$$\tau_i(\tilde{t}_i) = \text{marg}_{\Theta_0^* \times T_{-i}^*} \tilde{\mu}^i(\phi)$$

and

$$\tilde{\mu}^i(\phi)[\Theta_0 \times \{(t_{-i}, s_{-i}) : \text{ISR}_{-i}^k(t_{-i}) = \{s_{-i}\}\}] = 1.$$

Since  $\tilde{T}_{-i}(s_i) \subseteq \text{supp}(\text{marg}_{T_{-i}^*} \mu^{s_i})$  by hypothesis, we have that (from the definition of  $\tilde{T}_{-i}(s_i)$ )  $\bigcup_{t_{-i} \in \tilde{T}_{-i}(s_i)} \text{ISR}_{-i}^k(t_{-i}) = S_{-i}$ . Hence the conditional conjectures are uniquely determined for all  $h \in \mathcal{H}$ . These conjectures are given by  $\tilde{\mu}^i(\phi) = \tau_i(\tilde{t}_i) \circ \varkappa^{-1}$ , with  $\varkappa$  defined as

$$\varkappa(\theta_0, (\lambda, \kappa^{s_{-i}, \lambda})) = (\theta_0, (\lambda, \kappa^{s_{-i}, \lambda}), s_{-i}).$$

Furthermore, for each  $h$ ,

$$\text{marg}_{\Theta \times S_{-i}} \tilde{\mu}^i(h) = \text{marg}_{\Theta \times S_{-i}} \mu^{s_i}(h).$$

To see this, given the observation that  $\text{supp}(\text{marg}_{S_{-i}} \mu^{s_i}(\phi)) = S_{-i}$ , it suffices to show that  $\text{marg}_{\Theta \times S_{-i}} \tilde{\mu}^i(\phi) = \text{marg}_{\Theta \times S_{-i}} \mu^{s_i}(\phi)$ . But this is immediate, given that from the definition of  $\varkappa$  and  $\varphi$ , we have

$$\text{proj}_{\Theta_0 \times L_{-i}^{k-1} \times S_{-i}}^{-1} \circ \varkappa \circ \varphi = I.$$

( $I$  is the identity map.) Hence,  $\tilde{\mu}^i$  is uniquely determined for all  $h$  and it is equal to  $\mu^{s_i}$ , which makes  $s_i$  the unique best response. Hence  $\text{ISR}_i^{k+1}(\tilde{t}_i) = \{s_i\}$ .

The proof of statement (iii) in the lemma is identical to WY's: Define

$$\tilde{T}_i^{\tilde{t}_i} = \{\tilde{t}_i\} \cup \left( \bigcup_{(\theta, t_{-i}^{s_{-i}}) \in \text{supp}(\tau_i(\hat{t}_i))} T_i^{t_{-i}^{s_{-i}}} \right),$$

$$\tilde{T}_j^{\tilde{t}_i} = \bigcup_{(\theta, t_{-i}^{s_{-i}}) \in \text{supp}(\tau_i(\hat{t}_i))} T_{-i}^{t_{-i}^{s_{-i}}} \quad \text{for } j \neq i.$$

*Q.E.D.*

Given the lemmata above, the proof of Theorem 1 is immediate:

PROOF OF THEOREM 1: Take any  $\hat{t} \in \hat{T}$  and any  $s \in \text{ISR}(\hat{t})$ . For each  $i$ , from Lemma 3 there exists a sequence  $\{t_i^m\} \subseteq \hat{T}_{i,\theta}$  of finite types such that  $t_i^m \rightarrow \hat{t}_i$  and for each  $i$ ,  $s_i \in \text{SSR}_i(t_i^m)$  for each  $m$ , for conjectures  $\mu^{s_i}$  as in the thesis of Lemma 3 and in the hypothesis of Lemma 4. Then we can apply Lemma 4 to the types  $t_i^m$  for each  $m$ : for  $s_i \in \text{SSR}_i(t_i^m)$ , for each  $k$ , there exists a sequence  $\{\tilde{t}_i^{m,k}\}_{k \in \mathbb{N}}$  such that  $\tilde{t}_i^{m,k} \rightarrow t_i^m$  for  $k \rightarrow \infty$  such that  $\text{ISR}_i(\tilde{t}_i^{m,k}) = \{s_i\}$ . Because the universal type space is metrizable, there exists a sequence  $k_m \rightarrow \infty$  with  $t_i^{m,k_m} \rightarrow \hat{t}_i$ . Set  $\hat{t}_i^m = t_i^{m,k_m}$ , so that  $\hat{t}^m \rightarrow \hat{t}$  as  $m \rightarrow \infty$  and  $\text{ISR}(\hat{t}^m) = \{s\}$  for each  $m$ . Q.E.D.

B. Proof of Proposition 3

The first part follows trivially from the fact that the assumptions on information only play a role at unexpected histories. For the second part, it suffices to show that for any  $\langle \Theta_0, (\Theta_i, u_i)_{i \in N} \rangle$  embedded in  $\langle \Theta_0^*, (\Theta_i^*, u_i^*)_{i \in N} \rangle$ , for any  $\mathcal{T}_\theta$  and for any  $t_i \in T_{i,\theta}$ ,  $\text{ISR}_i^{\mathcal{T}_\theta}(t_i) = \text{ISR}_i^{\mathcal{T}_{\theta^*}}(\kappa_i^*(t_i))$ . The proof is by induction. Let  $t_i^* = \kappa_i^*(t_i)$ . Clearly, if  $\theta_i(t_i) = \hat{\pi}_i^0(t_i^*)$ ,  $\text{ISR}_i^{\mathcal{T}_\theta,1}(t_i) = \text{ISR}_i^{\mathcal{T}_{\theta^*},1}(t_i^*)$ . As the inductive hypothesis, assume that  $(\hat{\pi}_i^n(t_i))_{n=0}^{k-1} = (\hat{\pi}_i^n(t_i^*))_{n=0}^{k-1}$  implies that  $\text{ISR}_i^{\mathcal{T}_\theta,k}(t_i) = \text{ISR}_i^{\mathcal{T}_{\theta^*},k}(t_i^*)$ , and suppose that  $(\hat{\pi}_i^n(t_i))_{n=0}^k = (\hat{\pi}_i^n(t_i^*))_{n=0}^k$ . We show that  $(\hat{\pi}_i^n(t_i))_{n=0}^k = (\hat{\pi}_i^n(t_i^*))_{n=0}^k$  implies that  $\text{ISR}_i^{\mathcal{T}_\theta,k+1}(t_i) = \text{ISR}_i^{\mathcal{T}_{\theta^*},k+1}(t_i^*)$ . Under the inductive hypothesis,  $s_{-i} \in \text{ISR}_{-i}^{\mathcal{T}_\theta,k}(t_{-i})$  for some  $t_{-i} \in \text{supp } \tau_i(t_i)$  if and only if  $s_{-i} \in \text{ISR}_{-i}^{\mathcal{T}_{\theta^*},k}(t_{-i}^*)$  for some  $t_{-i}^* \in \text{supp } \tau_i(t_i^*)$ . In private values environments, only the conjectures about  $S_{-i}$  are payoff relevant for player  $i$  ( $\theta_{-i}$ 's do not affect  $i$ 's payoffs, and  $\Theta_0$  is a singleton). Thus, under the inductive hypothesis,  $\exists \mu^i \in \Phi_i(t_i)$  such that  $\text{supp}(\text{marg}_{T_{-i,\theta} \times S_{-i}} \mu^i) \subseteq \text{ISR}_{-i}^{\mathcal{T}_\theta,k+1}$  and  $s_i \in r_i(\mu^i|t_i)$  if and only if  $\exists \hat{\mu}^i \in \Phi_i(t_i)$  such that  $\text{supp}(\text{marg}_{T_{-i,\theta^*} \times S_{-i}} \hat{\mu}^i) \subseteq \text{ISR}_{-i}^{\mathcal{T}_{\theta^*},k+1}$  and  $s_i \in r_i(\hat{\mu}^i|t_i^*)$ . (Remember, the only restrictions on the conjectures over  $S_{-i}$  imposed by ISR are at the beginning of the game.) Hence  $\text{ISR}_i^{\mathcal{T}_\theta,k+1}(t_i) = \text{ISR}_i^{\mathcal{T}_{\theta^*},k+1}(t_i^*)$ . Q.E.D.

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